Metric Spaces, Entropic Spaces and Convexity

WORK IN PROGRESS

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Introduction

Lawvere: state spaces for thermodynamics should be certain "metric-like" enriched categories

entropic spaces

Baez-Lynch-Moeller: state spaces and entropy should be convex spaces & concave maps to $[-\infty, +\infty]$.

GOAL: Synthesise these to get a category of convex entropic spaces & concave maps.

Inspiration: Fritz-Perrone approach to convexity monad on metric spaces.
(Also Mardare-Panangaden-Plotkin.)

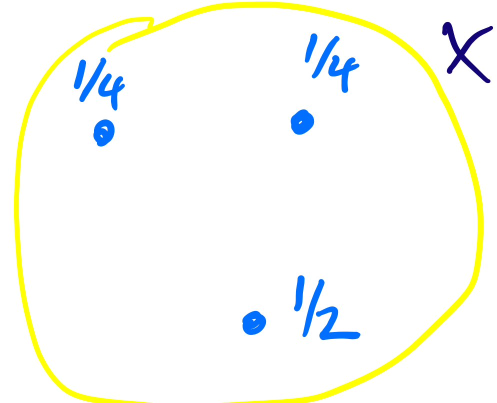
I Convexity Monads

$C : \mathbf{Set} \to \mathbf{Set}$ monad

$$C(X) = \left\{ \sum_{i=1}^{n} \alpha_i \ulcorner x_i \urcorner \;\middle|\; \alpha_i \in [0,1],\ \sum \alpha_i = 1,\ x_i \in X,\ n \in \mathbb{N} \right\}$$

formal symbol

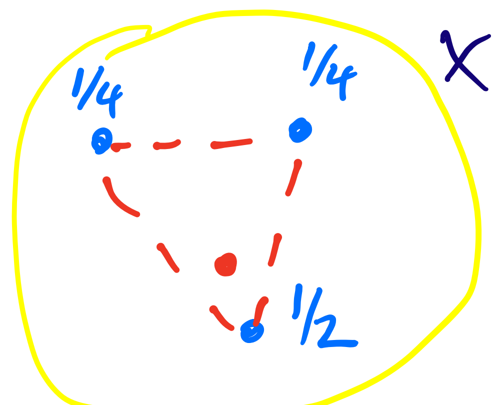
"Convexity monad", "distribution monad", "finitely supported measures monad".



Algebras for C: convex spaces

$$C(X) \xrightarrow{e_X} X$$

$$\sum \alpha_i \ulcorner x_i \urcorner \longmapsto \sum \alpha_i x_i$$



Algebra map $f : X \to Y$

$$f\left(\sum \alpha_i x_i\right) = \sum \alpha_i f(x_i)$$ convex linear maps

Rational convexity monad

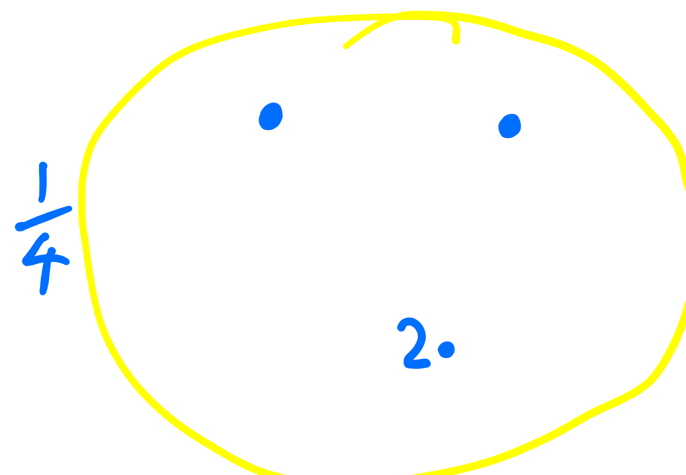
$$C_{\mathbb{Q}} : \mathrm{Set} \to \mathrm{Set} \qquad \text{(submonad)}$$

$$C_{\mathbb{Q}}(X) = \left\{ \sum_{i=1}^{n} \alpha_i \ulcorner x_i \urcorner \;\middle|\; \alpha_i \in [0,1] \cap \mathbb{Q},\ \sum \alpha_i = 1,\ x_i \in X,\ n \in \mathbb{N} \right\}$$

If $c \in C_{\mathbb{Q}}(X)$ then $c = \frac{1}{N} \sum' \ulcorner x_i \urcorner$ for some N.

eg $\quad \frac{1}{4}\ulcorner x_1 \urcorner + \frac{1}{4}\ulcorner x_2 \urcorner + \frac{1}{2}\ulcorner x_3 \urcorner = \frac{1}{4}\left(\ulcorner x_1 \urcorner + \ulcorner x_2 \urcorner + \ulcorner x_3 \urcorner + \ulcorner x_3 \urcorner\right)$

This has a more discrete feel.



II. Enriched Categories

We will enrich over a commutative quantale

(aka (co)complete, skeletal, closed sym. monoidal thin category)

Concentrate on $\overline{\mathbb{R}}_+ = (([0,\infty], \geq), +, 0)$

Also of interest $\overline{\mathbb{R}} = (([-\infty, +\infty], \geq), +, 0)$ convex analysis

$\overline{\mathbb{R}}^0 = ([-\infty, +\infty], \leq), +, 0)$ entropic spaces

$\mathrm{Truth} = ((\{T, F\}, \vdash), \&, T)$ preorders

$\overline{\mathbb{R}}_+$-category X:

$X(a,b) \in [0,\infty] \quad \forall a,b \in X$

$X(a,b) + X(b,c) \geq X(a,b) \quad \forall a,b,c \in X$

$0 \geq X(a,a) \quad \forall a \in X$

(Lawvere) metric space versus classical metric space

Not necessarily symmetric, can have ∞ value.

Get a category of metric spaces

$\overline{\mathbb{R}}_+$-Cat

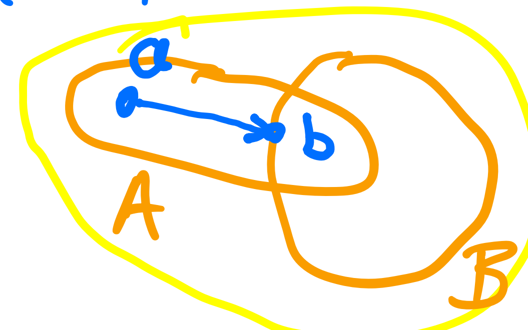
morphisms: short maps – distance non-increasing

<u>Examples</u> (i) $\overline{\mathbb{R}}_+$ is an $\overline{\mathbb{R}}_+$-category: $\overline{\mathbb{R}}_+(a,b) = b \dot{-} a$

$$:= \max(b-a,0)$$

(ii) M (classical) metric space $S_M = \{\text{compact subsets of } M\}$

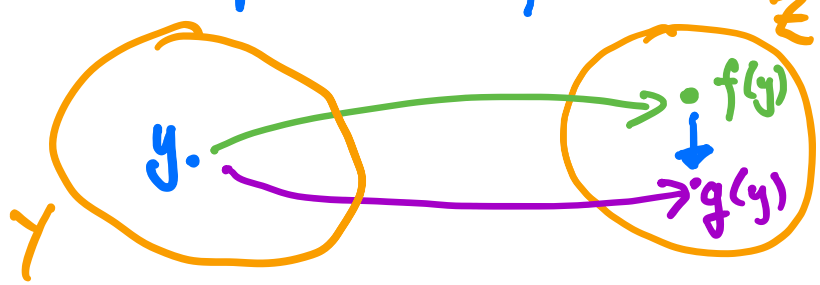
$$S_M(A,B) = \sup_a \inf_b M(a,b)$$



$$S_M(A,B) = 0 \iff A \subseteq B$$

(iii) Y, Z $\overline{\mathbb{R}}_+$-cats then $[Y,Z] = \{\text{short maps } Y \to Z\}$

$$[Y,Z](f,g) = \sup_{y \in Y} Z(f(y), g(y))$$

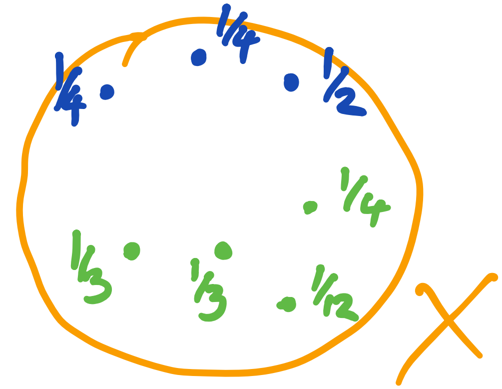


III. Convex Metric Spaces

Want a monad

$C : \bar{\mathbb{R}}_+\text{-cat} \to \bar{\mathbb{R}}_+\text{-cat}; \quad X \mapsto C(X)$

But what metric on $C(X)$?



Can consider $\sum \alpha_i \ulcorner x_i \urcorner \in C(X)$ as a finite measure
ie a function on functions

$$\sum \alpha_i \ulcorner x_i \urcorner \mapsto \left(f \mapsto \sum \alpha_i f(x_i) \right)$$ ← short map

$$C(X) \to \left[[X, \bar{\mathbb{R}}_+], \bar{\mathbb{R}}_+ \right]$$ ← map of sets

Get metric

$$C^{DD}\left(\sum \alpha_i \ulcorner x_i \urcorner, \sum \beta_j \ulcorner x'_j \urcorner\right) = \sup_{f \in [X, \bar{\mathbb{R}}_+]} \left(\sum \beta_j f(x'_j) \dot{-} \sum \alpha_i f(x_i) \right)$$

and whence a monad

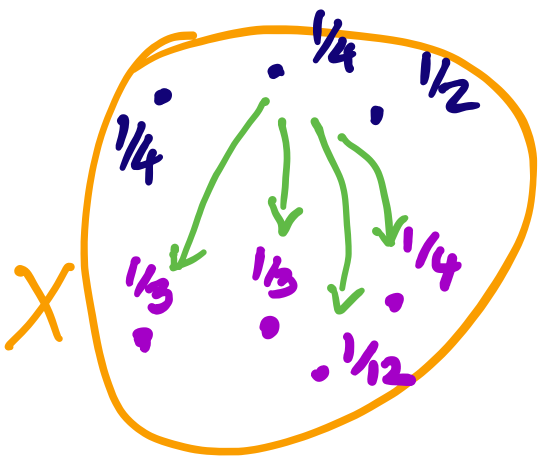
$$C^{DD} : \bar{\mathbb{R}}_+\text{-cat} \to \bar{\mathbb{R}}_+\text{-cat}$$

DD = Double Dual

There's another way to define a metric.

Think of optimal transport of goods.

$$C^{OT}\left(\sum \alpha_i \ulcorner x_i \urcorner, \sum \beta_j \ulcorner x_j' \urcorner\right) = \inf\left\{ \sum_{ij} A_{ij} X(x_i, x_j') \;\middle|\; A_{ij} \in [0,1],\ \sum_j A_{ij} = \alpha_i,\ \sum_i A_{ij} = \beta_j \right\}$$



A_{ij} is amount of goods $x_i \rightsquigarrow x_j'$

$A_{ij} X(x_i, x_j') =$ cost of transportation

C^{OT} is the minimum cost of transportation

$$C^{OT} : \overline{\mathbb{R}}_+\text{-Cat} \to \overline{\mathbb{R}}_+\text{-Cat}$$

(aka Wasserstein or Kantorovich-Rubinstein metric)

We have

$$C^{DD}(X) = C^{OT}(X)$$

if

X is classical
[Kantorovich duality]

X is symmetric
$X(a,b) = X(b,a)$
[Callum Reader]

X has finite distances
$X(a,b) < \infty$
[Me]

Likely in general,
I think!

Convex metric space is an algebra for $C^{\cdot\cdot}$

i.e. metric space and convex space with compatibility.

Can characterize this for

$$C^{OT}_{\mathbb{Q}} : \bar{\mathbb{R}}_+\text{-cat} \to \bar{\mathbb{R}}_+\text{-cat}$$

Compatibility is

$$\sum_i \alpha_i X(x_i, x_i') \geq X\left(\sum_i \alpha_i x_i, \sum_i \alpha_i x_i'\right)$$

$C^{OT}_{\mathbb{Q}}$ is simpler to handle as purely combinatorial!

Fritz-Perrone use this to characterize convexity for complete classical metric spaces

IV: 2-MONAD & CONCAVE MAPS

Every enriched category has an underlying category.

Every $\overline{\mathbb{R}}_+$-enriched category has an underlying preorder.

$$a \geq_X b \iff X(a,b) = 0$$

Eg (i) $\overline{\mathbb{R}}_+$: $\quad a \geq_{\overline{\mathbb{R}}_+} b \iff a \geq b$

(ii) S_M: $\quad A \geq_{S_M} B \iff A \subseteq B$

(compact subsets of M)

(iii) $[Y,Z]$: $\quad F \geq_{[Y,Z]} G \iff F(y) \geq_Z G(y) \quad \forall y \in Y$

(iv) X discrete $\quad X(a,b) = \begin{cases} 0 \\ \infty \end{cases}$

X classical

$$a \geq_X b \iff a = b$$

Convexity monad gives an enriched monad

$$C^{DD} : \overline{\mathbb{R}}_+\text{-}\underline{CAT} \to \overline{\mathbb{R}}_+\text{-}\underline{CAT}$$

($\mathbb{R}_+$-cat enriched)

This is because

$$[X, Y] \to [C(X), C(Y)]$$

is short.

Get a 2-monad

$$C^{DD} : \overline{\mathbb{R}}_+\text{-}CAT \to \overline{\mathbb{R}}_+\text{-}CAT$$

ob - metric spaces
mor - short maps
2-mor - $\geqslant$

Strict algebras: convex metric spaces $C^{DD}(X) \to X$

Lax algebra maps: $X \xrightarrow{f} Y$

$$\begin{tikzcd}
C^{DD}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\geqslant"] & X \arrow[d] \\
C^{DD}(Y) \arrow[r] & Y
\end{tikzcd}$$

ie $\sum \alpha_i f(x_i) \geqslant_Y f\left(\sum \alpha_i x_i\right)$

convex maps

Can replace $\overline{\mathbb{R}}_+$ by any convex quantale

Eg

$\overline{\mathbb{R}} = (([-\infty, +\infty], \geq), +, 0)$ convex analysis

$\overline{\mathbb{R}}^o = ([-\infty, +\infty], \leq), +, 0)$ entropic spaces

$\mathrm{Truth} = ((\{T, F\}, \vdash), \&, T)$ preorders $+ C^{DD} \neq C^{OT}$

Obtain 2-monad

$C : \overline{\mathbb{R}}^o\text{-}\mathrm{CAT} \to \overline{\mathbb{R}}^o\text{-}\mathrm{CAT}$

2-category of strict algebras, lax morphisms & $\geq$.

convex entropic spaces, concave maps & $\geq$