Magnitude of odd balls

Simon Willerton
University of Sheffield

Euler characteristics (some category theory!)

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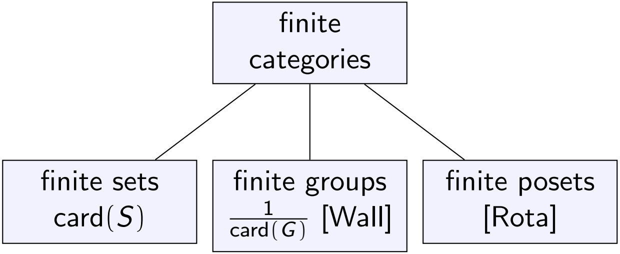

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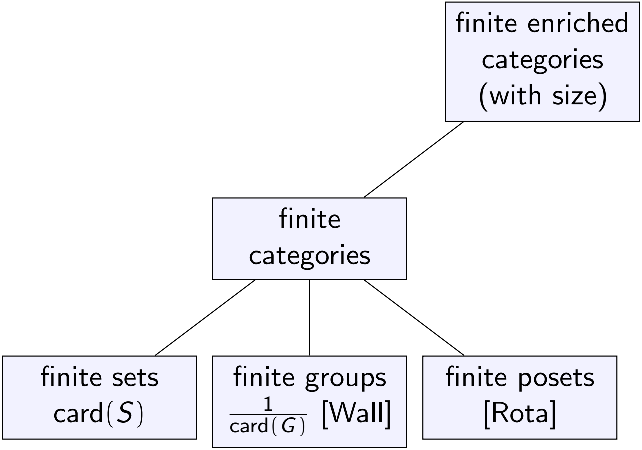

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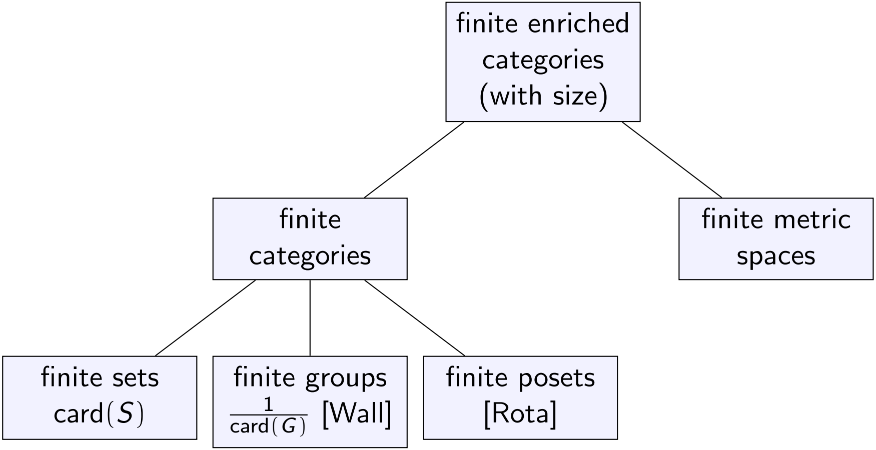

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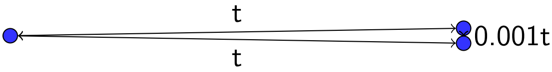
$tA :=$ 

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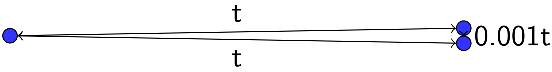

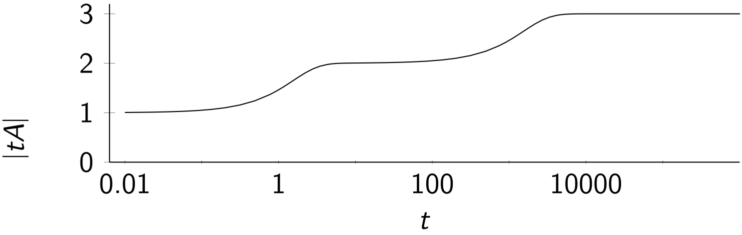

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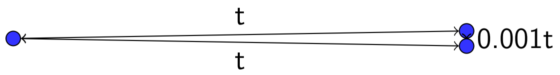
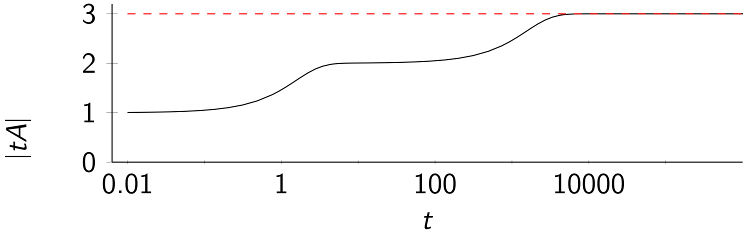
$tA :=$ 



If A has N points then $|tA| \to N$ as $t \to \infty$.

Extending to infinite metric spaces

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What happens when try to approximate an infinite subset of $\mathbb{R}^n$?

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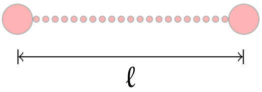

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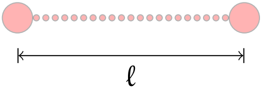

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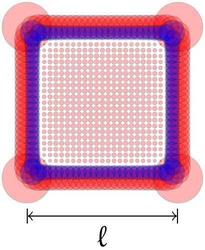

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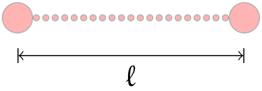

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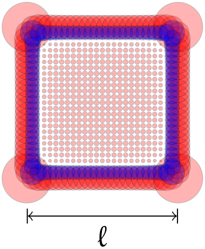

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Definition/Theorem. If $X \subset \mathbb{R}^n$ is compact and $A_n \to X$ in the Hausdorff topology then we can define $|X| := \lim_n |A_n|$.

Magnitude knows about things such as volume and Minkowski dimension.

Better definition of magnitude?

For X metric space, a weight measure is a signed measure on X such that

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Unfortunately these don't exist in general!

The problem is the limit of signed measures is not necessarily a measure.
For example, on $\mathbb{R}$ consider $(\mu_i)_{i=0}^{\infty}$ with $\mu_i = i(\delta_1 - \delta_{1-\frac{1}{i}})$.

$$\int_X f(x) \mathrm{d}\mu_i = \frac{f(1) - f(1 - \frac{1}{i})}{i} \to f'(1) \quad \text{as } i \to \infty.$$

The association $f \mapsto f'(1)$ is not the integration of a measure.
It is something more general: the evaluation of a distribution.

Distributions

A distribution on $\mathbb{R}^n$ is a linear functional on some suitable class of functions. Write $\langle w, f \rangle$ for the evaluation of a distribution w on a function f.
E.g.

(i) For each signed measure μ we have an associated distribution with

$$\langle \mu, f \rangle := \int_{\mathbb{R}^n} f \, \mathrm{d}\, \mu.$$

(ii) For a cooriented, smooth, codim 1 submanifold $\Sigma \subset \mathbb{R}^n$, and $i \in \mathbb{N}$

$$\langle w_i, f \rangle := \int_{\Sigma} \frac{\partial^i}{\partial \nu^i} f(\mathbf{x}) \, \mathrm{d}\, \mathbf{x},$$

where $\frac{\partial}{\partial \nu}$ means derivative in the normal direction to the submanifold.

Weight distributions

Suppose $X \subset \mathbb{R}^n$ is compact. A weight distribution w is a distribution supported on X such that

$$\langle w, e^{-\mathrm{d}(\mathbf{s},\cdot)} \rangle = 1 \quad \text{for every } \mathbf{s} \in X.$$

The magnitude of X is given by $|X| = \langle w, \mathbf{1} \rangle$.

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Try to calculate the magnitude of a non-trivial space!
Guess a weight distribution for B_R^n, the radius R ball of dimension $n = 2p+1$.

$$\langle w, f \rangle = \frac{1}{n!\,\omega_n} \left(\int_{\mathbf{x} \in B_R^n} f \,\mathrm{d}\mathbf{x} + \sum_{i=0}^{p} \beta_i(R) \int_{\mathbf{x} \in S_R^{n-1}} \frac{\partial^i}{\partial \nu^i} f \,\mathrm{d}\mathbf{x} \right)$$

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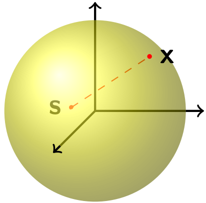
Need to solve the weight equation for every $\mathbf{s} \in B_R^n$ to find $(\beta_i(R))_{i=0}^{p}$.
Then

$$|B_R^n| = \tfrac{1}{n!} \left(R^n + n\beta_0(R) R^{n-1} \right).$$

The Key Integral

$$\frac{1}{n!\,\omega_n}\int_{\mathbf{x}\in S_R^{n-1}} e^{-|\mathbf{x}-\mathbf{s}|}\,\mathrm{d}\mathbf{x} \quad \text{for } \mathbf{s}\in B_R^n$$



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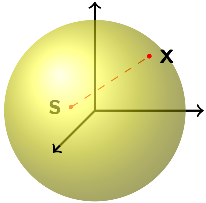

Theorem

For $n = 2p+1$, $R > 0$ and $s = |\mathbf{s}| < R$, then

$$\frac{1}{n!\,\omega_n}\int_{\mathbf{x}\in S_R^{n-1}} e^{-|\mathbf{x}-\mathbf{s}|}\,\mathrm{d}\mathbf{x} = \frac{(-1)^p e^{-R}}{2^p p!}\sum_{i=0}^{p}\binom{p}{i}\chi_{p+i}(R)\tau_i(s).$$

Reverse Bessel polynomials

$$\chi_0(R) = 1;$$
$$\chi_1(R) = R;$$
$$\chi_2(R) = R^2 + R;$$
$$\chi_3(R) = R^3 + 3R^2 + 3R$$
$$\chi_4(R) = R^4 + 6R^3 + 15R^2 + 15R.$$

modified spherical Bessel functions-ish

$$\tau_0(s) = \cosh(s);$$
$$\tau_1(s) = -\frac{\sinh(s)}{s};$$
$$\tau_2(s) = \frac{\cosh(s)}{s^2} - \frac{\sinh(s)}{s^3};$$
$$\tau_3(s) = -\frac{\sinh(s)}{s^3} + \frac{3\cosh(s)}{s^4} - \frac{3\sinh(s)}{s^5}$$

Solving the weight equations

Trying to solve the weight equation for every $s \in S_R^{n-1}$ gives a linear system.

$$\begin{pmatrix} \chi_p(R) & \delta\chi_p(R) & \dots & \delta^p\chi_p(R) \\ \chi_{p+1}(R) & \delta\chi_{p+1}(R) & \dots & \delta^p\chi_{p+1}(R) \\ \vdots & \vdots & & \vdots \\ \chi_{2p}(R) & \delta\chi_{2p}(R) & \dots & \delta^p\chi_{2p}(R) \end{pmatrix} \begin{pmatrix} \beta_0(R) \\ \beta_1(R) \\ \vdots \\ \beta_p(R) \end{pmatrix} = \begin{pmatrix} \chi_{p+1}(R)/R \\ \chi_{p+2}(R)/R \\ \vdots \\ \chi_{2p+1}(R)/R \end{pmatrix}$$

But remember the magnitude has the following form.

$$|B_R^n| = \frac{1}{n!}\left(R^n + n\beta_0(R)R^{n-1}\right),$$

So we can add this to our linear system.

$$\begin{pmatrix} \chi_p(R) & \delta\chi_p(R) & \dots & \delta^p\chi_p(R) & 0 \\ \chi_{p+1}(R) & \delta\chi_{p+1}(R) & \dots & \delta^p\chi_{p+1}(R) & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \chi_{2p}(R) & \delta\chi_{2p}(R) & \dots & \delta^p\chi_{2p}(R) & 0 \\ -nR^{n-1} & 0 & \dots & 0 & n! \end{pmatrix} \begin{pmatrix} \beta_0(R) \\ \beta_1(R) \\ \vdots \\ \beta_p(R) \\ \left|B_R^n\right| \end{pmatrix} = \begin{pmatrix} \chi_{p+1}(R)/R \\ \chi_{p+2}(R)/R \\ \vdots \\ \chi_{2p+1}(R)/R \\ R^n \end{pmatrix}$$

Now use Cramer's Rule...

The answer

$$|B_R^n| = \frac{\begin{vmatrix}\text{some matrix of} \\ \text{derivatives of } \chi_i(R)s\end{vmatrix}}{n!\begin{vmatrix}\text{some other matrix of} \\ \text{derivatives of } \chi_i(R)s\end{vmatrix}} = \cdots = \frac{\begin{vmatrix} \chi_2(R) & \chi_3(R) & \dots & \chi_{p+2}(R) \\ \chi_3(R) & \chi_4(R) & \dots & \chi_{p+3}(R) \\ \vdots & & \vdots & \\ \chi_{p+2}(R) & \chi_{p+3}(R) & \dots & \chi_{2p+2}(R) \end{vmatrix}}{n!\,R\begin{vmatrix} \chi_0(R) & \chi_1(R) & \dots & \chi_p(R) \\ \chi_1(R) & \chi_2(R) & \dots & \chi_{p+1}(R) \\ \vdots & & \vdots & \\ \chi_p(R) & \chi_{p+1}(R) & \dots & \chi_{2p}(R) \end{vmatrix}}$$

[Determinants with constant antidiagonals are called Hankel determinants.]

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$$\left|B_R^1\right| = R+1$$

$$\left|B_R^3\right| = \frac{R^3+6R^2+12R+6}{3!}$$

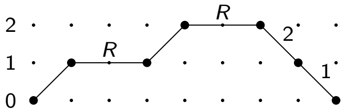
$$\left|B_R^5\right| = \frac{R^6+18\,R^5+135\,R^4+525\,R^3+1080\,R^2+1080\,R+360}{5!\,(R+3)}$$

$$\left|B_R^7\right| = \frac{R^{10}+40R^9+720R^8+\cdots+1814400R^2+1209600R+302400}{7!\,(R^3+12R^2+48R+60)}$$

Lots of things about these not (immediately) explained by the formula...

Combinatorial interpretation of reverse Bessel polynomials

A **length ℓ weighted Schröder path** is a lattice path from $(x, 0)$ to $(x + \ell, 0)$ of a certain form:



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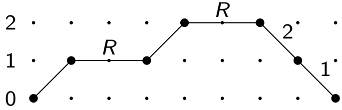

Theorem (Favreau/Sokal)

$$\chi_{i+1}(R)/R = \sum_{\gamma \text{ length } 2i} w(\gamma)$$

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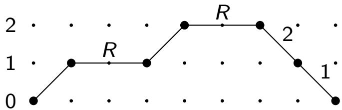

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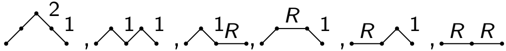
Example

$\chi_3(R)/R = \#\left\{ \right.$



$\left. \right\} = R^2 + 3R + 3$

Combinatorial interpretation of the determinants

Theorem (Rough version of Lindström-Gessel-Viennot Lemma)

- *Let G be a weighted, directed, acyclic graph.*
- *Suppose $\{K_i\}_{i=0}^k$ and $\{L_j\}_{j=0}^k$ be two sets of vertices in G.*
- *Let $M_{i,j}$ denote the weighted count of paths from K_i to L_j.*
- *Then subject to some condition on the vertices, the determinant*

$$\det[M_{i,j}]_{i,j=0}^k$$

  is the weighted count of all disjoint collections of $k+1$ paths joining K_i to L_i for $i = 0, \ldots, k$.

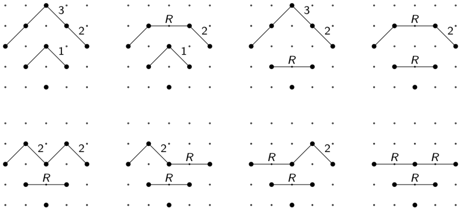
Each $\chi_i(R)$ is a count of lattice paths.

Corollary

- *The Hankel determinants $\det[\chi_{i+j+2}(R)]_{i,j=0}^p$ and $\det[\chi_{i+j}(R)]_{i,j=0}^p$ are counts of disjoint collections of lattice paths.*
- *Thus so are the numerator and denominator of $\left|B_R^n\right|$.*

Combinatorial interpretation of the determinants (ctd)

You end up with very nice expressions for the numerator and denominator. For example,

numerator $\left|B_R^3\right| = \#$ {  }

$$= R^3 + 6R^2 + 12R + 6$$

We now have a combinatorial interpretation of each of the coefficients:

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The payoff

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Theorem

- *Both numerator and denominator are monic polynomials of the obvious degrees with positive integer coefficients.*
- $\left|B_R^n\right| \to 1$ *as* $R \to 0$.
- $\left|B_R^n\right| = \frac{1}{n!}\left(R^n + \frac{n(n+1)}{2}R^{n-1} + \frac{(n-1)n(n+1)^2}{8}R^{n-2} + \dots\right)$ *as* $R \to \infty$.

Theorem (Gimperlein-Goffeng)

Suppose $X \in \mathbb{R}^n$ *is a smooth domain with* $n = 2m-1$ *then as* $R \to \infty$

$$|R \cdot X| \sim \frac{1}{n!\omega_n}\left(\mathrm{vol}(X)R^n + c_1\,\mathrm{vol}(\partial X)R^{n-1} + c_2\,TMC(\partial X)R^{n-2} + \cdots\right).$$