Looking at metric spaces as enriched categories

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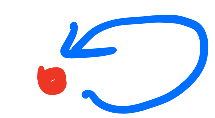
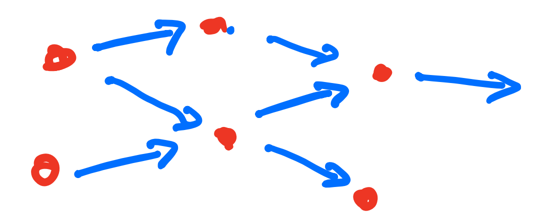
Enriched category theory.

"Large" categories: (algebraic) Vect, Set, Ab, Rep_A

(topological) Top, SSets,

"Small" categories:

- G a group, BG one object category 
- A an algebra, BA one object category 
- P a poset, considered as a category 

  $\exists!\, p \to q \iff p \leq q$
- X topological space, $\mathcal{O}_X$ poset of open sets

  $\exists!\, U \to V \iff U \subseteq V$

Given $\mathcal{C}, \mathcal{D}$ categories have

- product category $\mathcal{C} \times \mathcal{D}$
- functor category $[\mathcal{C}, \mathcal{D}] = \mathcal{D}^{\mathcal{C}}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "G"] \arrow[r, bend right=50, "F"'] \arrow[r, phantom, "\theta \Uparrow"] & \mathcal{D} \end{tikzcd}$$

objects: functors $\mathcal{C} \to \mathcal{D}$

morphisms: natural transformations

Category of "scalar valued functors" a.k.a (co-)presheaves

$$[\mathcal{C}, \mathrm{Set}] \quad \& \quad [\mathcal{C}^{op}, \mathrm{Set}]$$

contravariant functors

Eg - G group $\mathcal{C} = BG$

$$[BG, \mathrm{Set}] = \mathrm{Act}_G = \text{category of left } G\text{-actions and interwiners}$$

$$[BG^{op}, \mathrm{Set}] = {}_G\mathrm{Act} = \text{category of right } G\text{-actions}$$

- X topological space $\mathcal{C} = \mathcal{O}_X$

$$[\mathcal{O}_X^{op}, \mathrm{Set}] = \text{presheaves on } X$$

<u>Yoneda Lemma</u>

$$\mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \mathrm{Set}]$$

$$c \longmapsto (d \mapsto \mathcal{C}(d,c))$$

Think $\quad S \hookrightarrow \mathbb{Z}[S]$ (Set)

$$s \longmapsto \delta_s$$

Eg Cayley's Theorem

$\mathcal{C} = BG$

$$\bullet \circlearrowleft G \quad \hookrightarrow \quad G \bullet \circlearrowleft \mathrm{Set}(G,G)$$

Not always correct to use **Set** as the scalars!

Eg A an algebra then for A and Rep_A the hom-sets are vector spaces

$$\mathrm{Rep}_A = \{\text{linear functors } A \to \mathrm{Vect}\}$$

Want to do category theory over a different "base category".

Need: monoidal category $(\mathcal{V}, \otimes, \mathbb{I})$

$\mathcal{V}$ category, $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, $\mathbb{I} \in \mathrm{ob}\,\mathcal{V}$

Eg $(\mathrm{Set}, \times, \{*\})$, $(\mathrm{Vect}, \otimes, \mathbb{C})$, $(([0,\infty], \geq), +, 0)$

$\overline{\mathbb{R}}_+$

$a \to b$

Category	$\mathcal{V}$-enriched category
collection $\mathrm{ob}\,\mathcal{C}$	Collection $\mathrm{ob}\,\mathcal{C}$
• $\forall x,y$ $\quad \mathcal{C}(x,y) \in \mathrm{ob}\,\mathrm{Set}$	• $\forall x,y$ $\quad \mathcal{C}(x,y) \in \mathrm{ob}\,\mathcal{V}$
• $\forall x,y,z$ $\quad \mathcal{C}(x,y) \times \mathcal{C}(y,z) \xrightarrow{\text{function of sets}} \mathcal{C}(x,z)$	• $\forall x,y,z$ $\quad \mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \xrightarrow{\text{morphism in } \mathcal{V}} \mathcal{C}(x,z)$
• $\forall x$ $\quad id_x \in \mathcal{C}(x,x)$ $\{*\} \to \mathcal{C}(x,x)$	• $\forall x$ $\quad \mathbb{I} \to \mathcal{C}(x,x)$
+ axioms	+ axioms

$\mathcal{V}$-functor $\quad F : \mathcal{C} \to \mathcal{D}$ $\qquad F_0 : \mathrm{ob}\,\mathcal{C} \to \mathrm{ob}\,\mathcal{D}$ function

& $F_{xy} : \mathcal{C}(x,y) \to \mathcal{D}(F(x), F(y))$ in $\mathcal{V}$

If $\mathcal{V}$ is nice, then $\mathcal{V}$ is $\mathcal{V}$-category and can make $[\mathcal{C}, \mathcal{D}]$ into a $\mathcal{V}$-category

$\mathrm{Rep}_A = [BA, \mathrm{Vect}]$

$\overline{\mathbb{R}}_+$-enriched category

Collection ob X

- $\forall x, y$ $\quad X(x,y) \in [0,\infty]$

  $\forall x, y, z$ $\quad X(x,y) + X(y,z) \geq X(x,z)$

- $\forall x$ $\quad 0 \geq X(x,x)$ $\quad (0 = X(x,x))$

\+ no axioms

An $\overline{\mathbb{R}}_+$-category is a generalized metric space.

(i) Not necessarily symmetric (ii) Can have ∞ distance

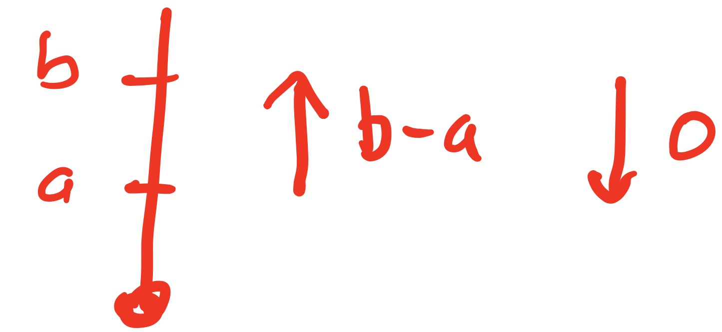
(iii) $d(x,y) = 0 \not\Rightarrow x = y$

An $\overline{\mathbb{R}}_+$-functor is a "short map" $f : X \to Y$, $X(x,x') \geq Y(f(x), f(x'))$

Examples

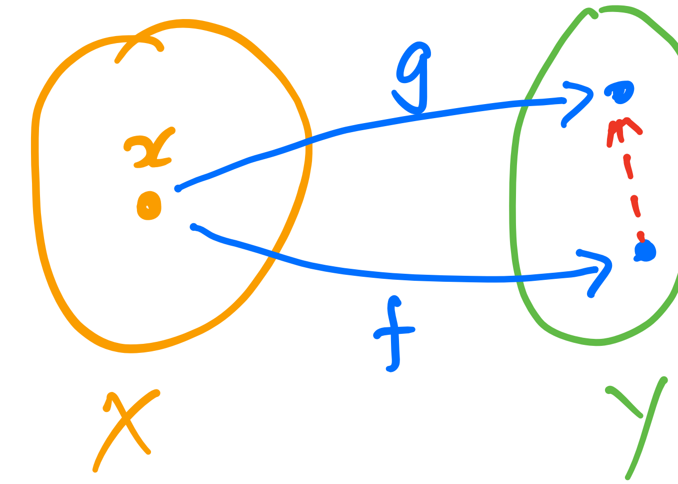
(i) $\overline{\mathbb{R}}_+$: $\quad \overline{\mathbb{R}}_+(a,b) = \max(b-a, 0)$

$=: b \dot{-} a$



(ii) $[X,Y]$:

$$[X,Y](f,g) = \sup_x Y(f(x), g(x))$$



$$[X,\overline{\mathbb{R}}_+](f,g) = \sup_x \left(g(x) \dot{-} f(x)\right)$$

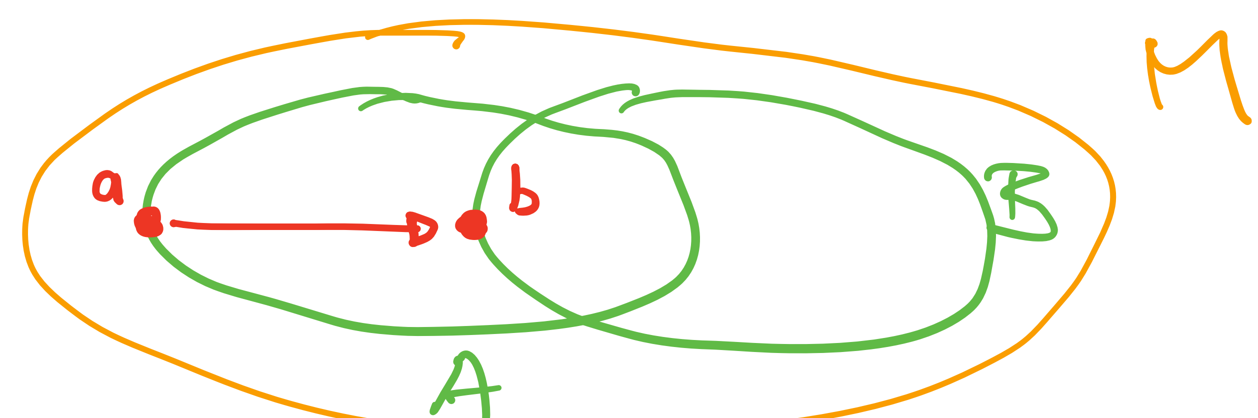
Yoneda: $\quad X \hookrightarrow [X^{op}, \overline{\mathbb{R}}_+], \quad x \mapsto (x' \mapsto d(x',x))$

Kuratowski isometric embedding

(iii) M (classical) metric space $\quad ob(S_M) = \{$compact subsets of $M\}$.

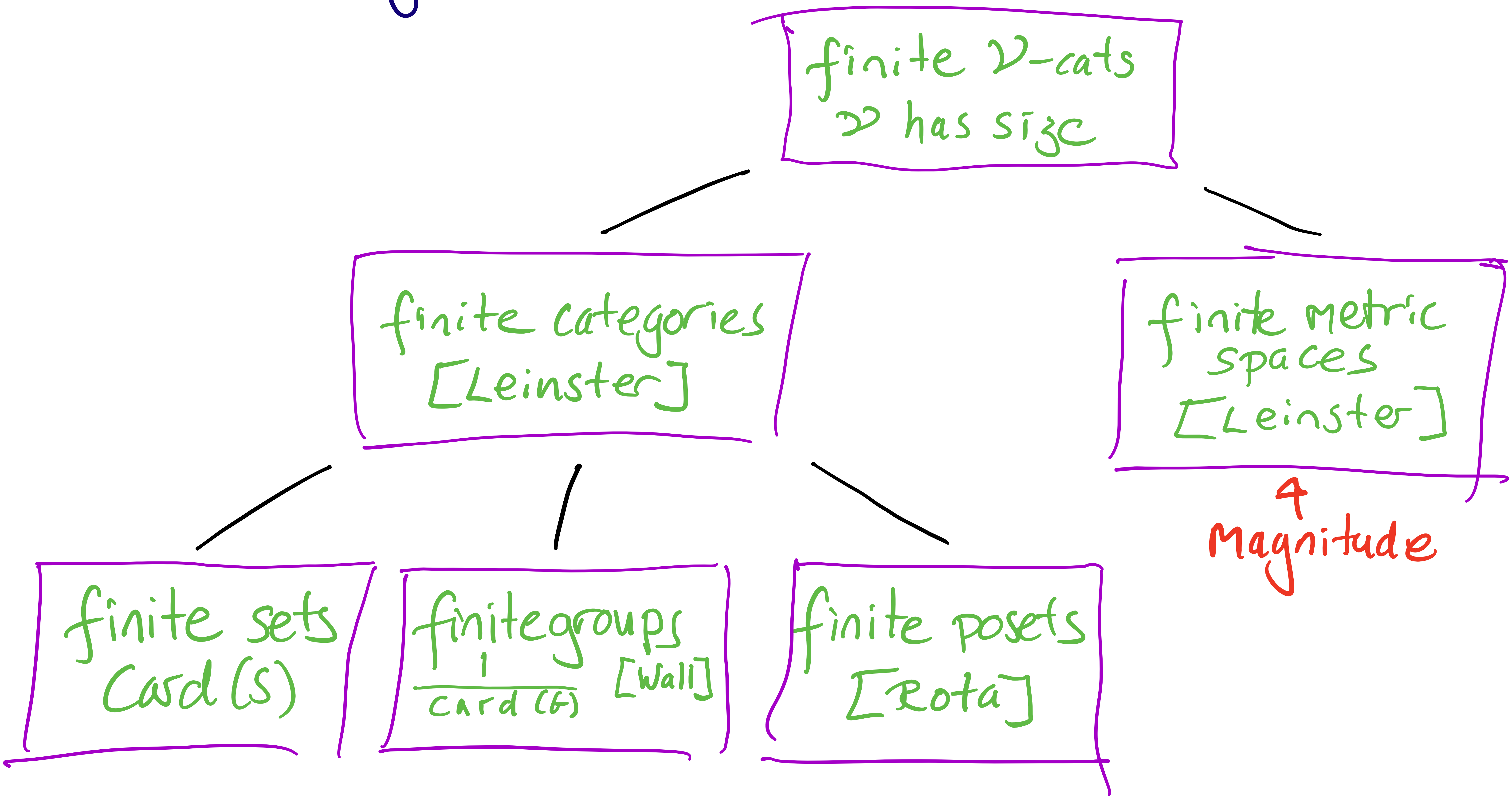
$$S_M(A,B) = \sup_{a\in A} \inf_{b\in B} M(a,b)$$

$S_M(A,B) = 0 \iff A \subseteq B$



Magnitude

Generalizing notions of Euler characteristic



Suppose X finite metric space, a weighting is $w: X \to \mathbb{R}$ such that

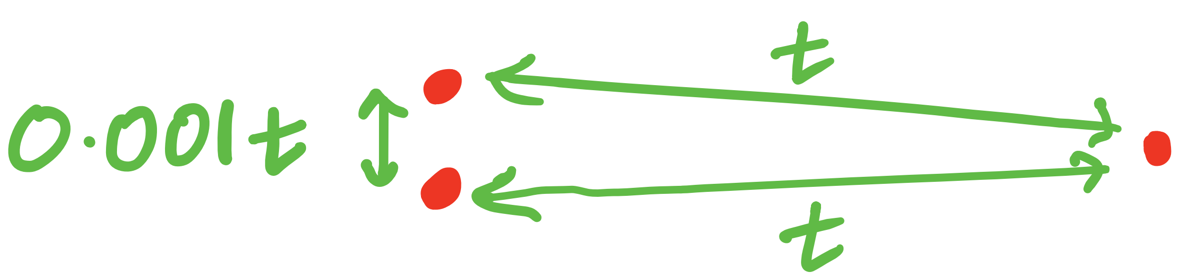
$$\sum_x e^{-X(x_0,x)} w(x) = 1 \quad \forall x_0 \in X$$

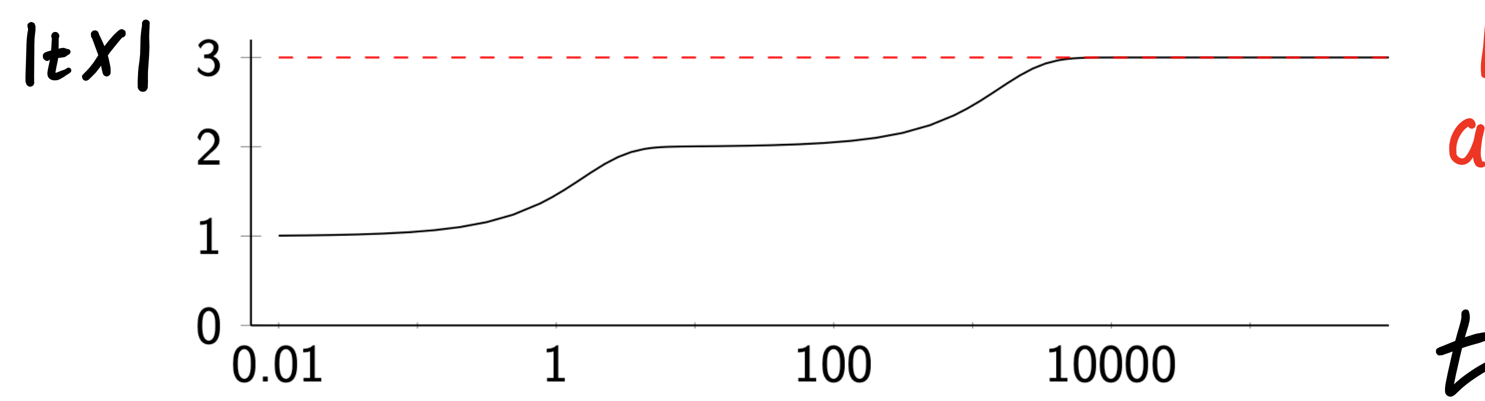
If a weighting exists, define magnitude

$$|X| := \sum_x w(x)$$

More interesting to consider magnitude function:

$$t \mapsto |tX| \quad \text{for } t > 0$$





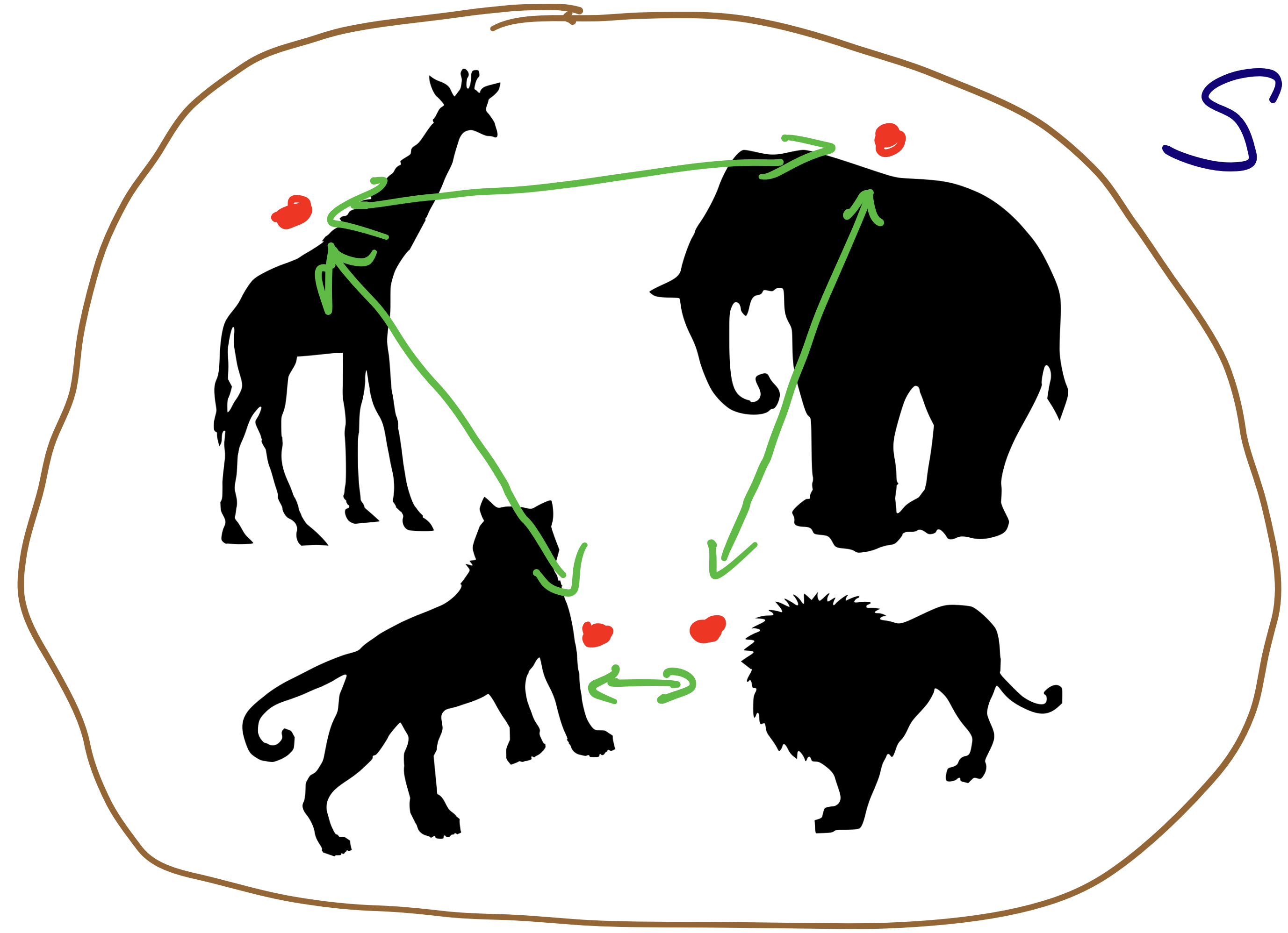
$|tX| \to \#X$ as $t \to \infty$

Magnitude measures "effective number of points"

Diversity

Model ecosystem as a set of species with a metric measuring 'difference' of species: $|S|$ is a measure of biodiversity.

[Solow-Polasky 1994]



Traditional diversity measures use the relative proportions of species – related to notions of entropy, Renyi, Shannon etc.

[Leinster-Cobbold] combined the two approaches to give a family of diversity indices using difference & relative proportion.

Magnitude $|S|_+$ gives maximum possible diversity.

The Leinster and Cobbold indices improve inferences about microbial diversity

Stavros D. Veresoglou [a, b], Jeff R. Powell [c], John Davison [d], Ylva Lekberg [e], Matthias C. Rillig [a, b]

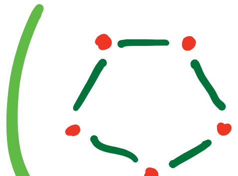
Khovanov homology: $\exists$ bigraded homology theory $KH_{\bullet,\bullet}$ of links such that

$$\mathrm{Jones}(L) = \sum_{i,j} (-1)^i q^j \, \mathrm{rk}\left(KH_{i,j}(L)\right)$$

polynomial in $q^{\pm 1}$ $\qquad$ $\chi(KH(L))$

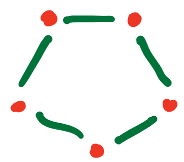
$\exists$ similar categorification of magnitude!

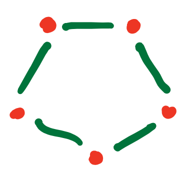
Simple example when $\mathcal{V} = ((\mathbb{N}, \geq), +, 0)$, restrict to graphs

$M := |t \cdot|_{q = e^{-t}} : \mathrm{Graphs} \to \mathbb{Z}[\![q]\!]$ $\qquad$ $M\Big($  $\Big) = 5 - 10q + 10q^2 - 20q^4 \ldots$

[Willerton - Hepworth] $\exists$ bigraded homology theory $MH_{\bullet\bullet}$ of graphs such that $M(G) = \chi(MH_{\bullet\bullet}(G))$

[Leinster - Shulman] $\exists$ bigraded homology theory of finite metric spaces that recovers magnitude as its Euler characteristic.

$M\Big($  $\Big) = 5 - 10q + 10q^2 - 20q^4 + 40q^5 - 40q^6 - 80q^8 + \dots$

$MH_{k,l}\Big($  $\Big)$

l \ k	0	1	2	3	4	5	6	7	8	9	10	11
0	5											
1		10										
2			10									
3			10	10								
4				30	10							
5					50	10						
6					20	70	10					
7						80	90	10				
8							180	110	10			
9							40	320	130	10		
10								200	500	150	10	
11									560	720	170	10

Magnitude for infinite metric spaces

$$L_n = \underbrace{\overbrace{\bullet \;\; \bullet \;\; \bullet \;\; \bullet \;\; \cdots \;\; \bullet \;\; \bullet}^{\ell}}_{n \text{ points}} \qquad \text{as } n \to \infty \quad |L_n| \to \frac{\ell}{2} + 1$$

[Meckes] For X compact $|X| := \sup_{A \subset X} |A|$.

If $X \subseteq \mathbb{R}^m$ & $A_n \to X$ then $|A_n| \to X$.

Growth rate of $|tX|$ is Minkowski dimension.

[Carbery-Barcelo & Willerton] $$|tB^3| = \frac{R^3 + 6R^2 + 12R + 6}{3!}$$

$|tB^{2n+1}|$ is ratio of determinants of Bessel polynomials

$|tB^{2n}|$ unknown in general!

[Goffeng-Gimperlein-Louca] If X^n is Riem. mfd with bdry

as $t \to \infty$ $$|tX| \sim \frac{1}{n!\,\omega_n}\left(\mathrm{vol}(X)\, t^n + \frac{n+1}{2}\mathrm{vol}(\partial X)\, t^{n-1} + \dots\right)$$

Tight span

Enriched Isbell completion

$$
\begin{tikzcd}
{[\mathcal{C}^{op}, \mathcal{V}]} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & {[\mathcal{C}, \mathcal{V}]^{op}} \arrow[ll, bend left, "R"] \\
& \mathcal{C} \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$f \in$ $\qquad\qquad\qquad\qquad$ $\ni g$

$$\mathcal{I}(\mathcal{C}) := \mathrm{Fix}(L, R)$$

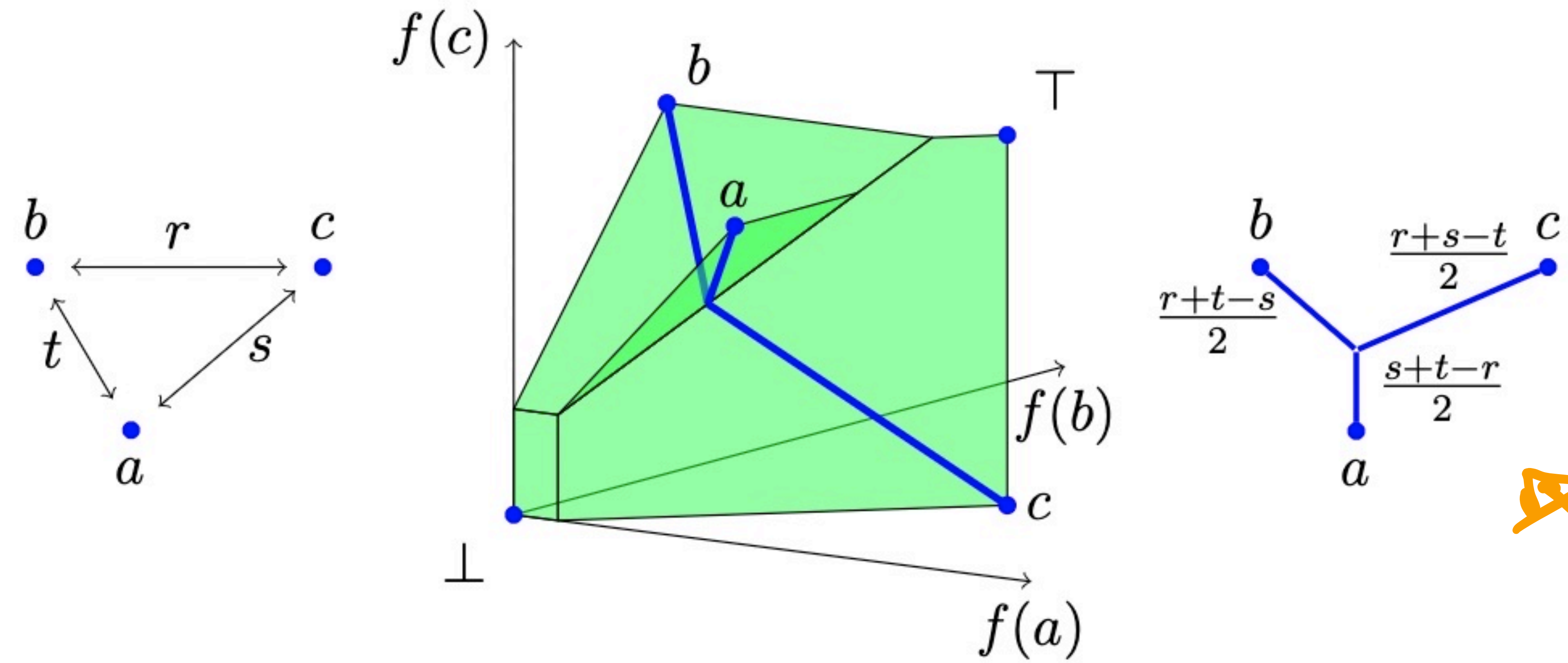
$$= \left\{ (f, g) \in [\mathcal{C}^{op}, \mathcal{V}] \times [\mathcal{C}, \mathcal{V}]^{op} \;\middle|\; \begin{array}{l} L(f) = g \\ f = R(g) \end{array} \right\}$$

$(f, g) \in \mathcal{I}(\mathcal{C})$ is a generalized element of $\mathcal{C}$

$$\mathcal{I}(\mathcal{C})((f,g), c) = g(c) \;;\quad \mathcal{I}(\mathcal{C})(c, (f,g)) = f(c)$$

Take $\mathcal{V} = \overline{\mathbb{R}}_+$ $\qquad [X^{op}, \overline{\mathbb{R}}_+] \subseteq \overline{\mathbb{R}}_+^{\#X}$

Eg



classical X

Tight span of X

Tight span / convex hull / injective envelope rediscovered many times

For general X have $I(X)$ is "directed tight span"

Also
- Kemajou - Künzi - Olela Otafudu
- Hirai - Koichi (multi commodity flow)

$\sim$ Develin - Sturmfels (tropical algebra).