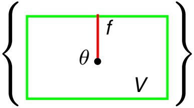
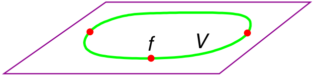
Two 2-traces

Simon Willerton
University of Sheffield

$\mathrm{Tr}^{\searrow}(f) :=$



$\mathrm{Tr}^{\circlearrowleft}(f) :=$



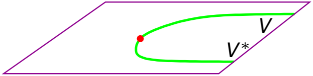
Traces

What is a trace?

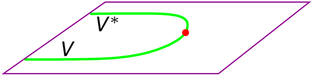
$$\begin{aligned} \mathrm{Tr}(f \circ g) &= \mathrm{Tr}(g \circ f) \\ \mathrm{Tr}(f) &= \mathrm{Tr}(a \circ f \circ a^{-1}) \end{aligned}$$

Traces in a monoidal category

In $(\mathcal{C}, \otimes, \mathbf{1})$, an object V^* is left-dual to V if there exist morphisms

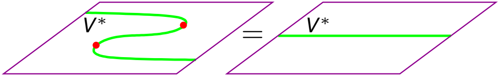


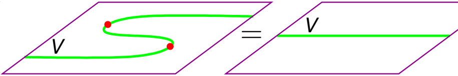
$\mathbf{1} \xleftarrow{\text{ev}} V^* \otimes V$



$V \otimes V^* \xleftarrow{\text{coev}} \mathbf{1}$

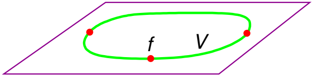
such that





If V is also left dual to V^* then V and V^* are bidual.

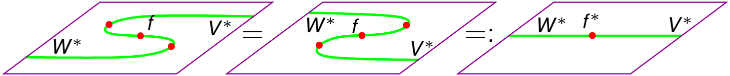
If V has a bidual and $V \xleftarrow{f} V$ define

$\mathrm{Tr}(f) :=$  $\in \mathrm{Hom}(\mathbf{1}, \mathbf{1})$.

In $(\mathrm{Vect}, \otimes, \mathbb{C})$ this gives the usual trace on finite dimensional vector spaces.

Transposes (or adjoints or duals)

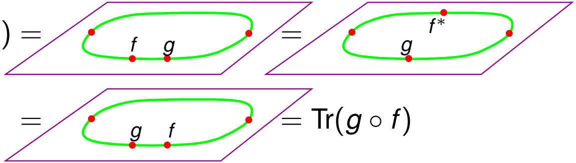
If V and W have biduals then $V \xleftarrow{f} W$ has a transpose (or is cyclic) if



Theorem (Trace property)

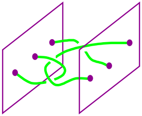
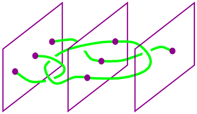
If $V \xleftarrow{f} W$ *and* $W \xleftarrow{g} V$ *with* f *having a transpose then*

$\mathrm{Tr}(f \circ g) =$



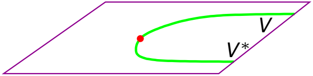
$= \mathrm{Tr}(g \circ f)$

Examples of monoidal bicategories

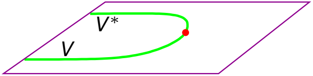
	objects	1-morphisms	composition	2-morphisms	
Span	Sets	$\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd}$	$\begin{tikzcd} & & T \times_Y S \arrow[dl] \arrow[dr] & & \\ & T \arrow[dl] \arrow[dr] & & S \arrow[dl] \arrow[dr] & \\ Z & & Y & & X \end{tikzcd}$	$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr] & T' \arrow[dl] \arrow[d] \\ Y & X \end{tikzcd}$	$\times$
Bim	Algebras/$\mathbb{C}$	${}_BM_A$	${}_CN_B \otimes_B {}_BM_A$	$\mathrm{Hom}_{B,A}({}_BM_A, {}_BM'_A)$	$\otimes_{\mathbb{C}}$
$\mathcal{V}$-Mod	$\mathcal{V}$-cats	$\mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathcal{V}$	$\otimes_D$	$\mathcal{V}$-nat trans	$\otimes$
2-Tang	pts in plane			cobordisms	$\coprod$
Var	$\mathbb{C}$-manifolds	$\begin{tikzcd} \mathcal{E}^\bullet \arrow[d] \\ Y \times X \end{tikzcd}$	convolution	$\mathrm{Ext}^\bullet_{Y \times X}(\mathcal{E}^\bullet, \mathcal{F}^\bullet)$	$\times$
DBim	Diff algs/$\mathbb{C}$	$\to {}_BM^i_A \to {}_BM^{i-1}_A \to$	$\otimes^{\mathrm{L}}_B$	$\mathrm{Ext}^\bullet_{B \times A^{\mathrm{op}}}({}_BM^\bullet_A, {}_BN^\bullet_A)$	$\otimes_{\mathbb{C}}$

Biduals in a monoidal bicategory

In $\mathcal{C}$, an object V^* is left-dual to V if there exist 1-morphisms

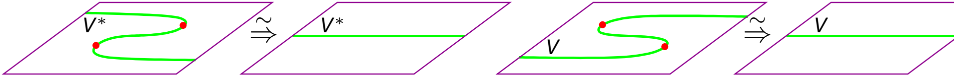


$$\mathbf{1} \xleftarrow{\mathrm{ev}} V^* \otimes V$$

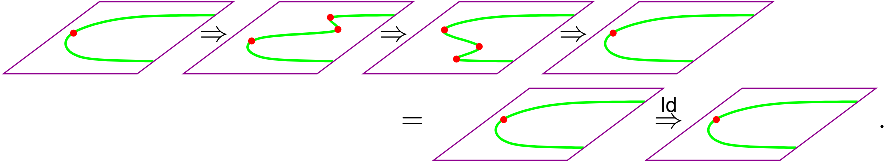


$$V \otimes V^* \xleftarrow{\mathrm{coev}} \mathbf{1}$$

and 2-isomorphisms



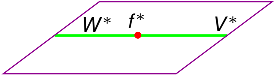
such that the Swallowtail Relations hold, e.g.,



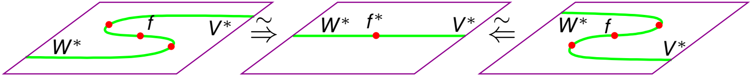
If V is also left dual to V^* then V and V^* are bidual.

Transposes in monoidal bicategories

A 1-morphism $V \xleftarrow{f} W$ has a transpose (or is cyclic) if there is a 1-morphism $W^* \xleftarrow{f^*} V^*$:

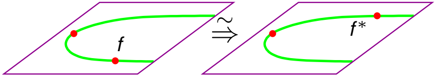


together with isomorphisms

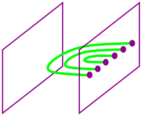
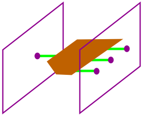
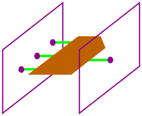


satisfying some conditions.

This gives for example



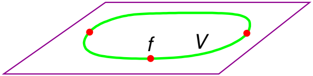
Examples of duals in monoidal bicategories

	object	bidual	evaluation	morphism	transpose
Span	X	X	$\star \leftarrow X \xrightarrow{\Delta} X \times X$	$Y \leftarrow T \rightarrow X$	$X \leftarrow T \rightarrow Y$
Bim	A	A^{op}	${}_{\mathbb{C}}A_{A\otimes A^{\mathrm{op}}}$	${}_{B}M_{A}$	${}_{A^{\mathrm{op}}}M_{B^{\mathrm{op}}}$
$\mathcal{V}$-Mod	$\mathcal{C}$	$\mathcal{C}^{\mathrm{op}}$	$\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \otimes \star \xrightarrow{\mathrm{Hom}} \mathcal{V}$	$\mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathcal{V}$	$(\mathcal{D}^{\mathrm{op}})^{\mathrm{op}} \otimes \mathcal{C}^{\mathrm{op}} \to \mathcal{V}$
2-Tang					
Var	X	X	$\begin{array}{c}\mathcal{O}_{\Delta}\\ \downarrow\\ \star \times X \times X\end{array}$	$\begin{array}{c}\mathcal{E}^{\bullet}\\ \downarrow\\ Y \times X\end{array}$	$\begin{array}{c}\mathcal{E}^{\bullet}\\ \downarrow\\ X \times Y\end{array}$
DBim	$A^{\bullet}$	$A^{\bullet\,\mathrm{op}}$	${}_{\mathbb{C}}A^{\bullet}_{A^{\bullet}\otimes A^{\bullet\,\mathrm{op}}}$	${}_{B^{\bullet}}M^{\bullet}_{A^{\bullet}}$	${}_{A^{\bullet\,\mathrm{op}}}M^{\bullet}_{B^{\bullet\,\mathrm{op}}}$

The round trace

If V has a bidual and $V \xleftarrow{f} V$ define the round trace:

$$\mathrm{Tr}^{\circlearrowright}(f) :=$$



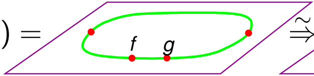
$$\in \text{1-Hom}(\mathbf{1}, \mathbf{1}).$$

Theorem (Trace property)

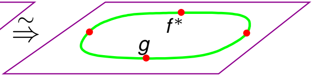
If $V \xleftarrow{f} W$ and $W \xleftarrow{g} V$ with f having a transpose then

$$\mathrm{Tr}^{\circlearrowright}(f \circ g) \cong \mathrm{Tr}^{\circlearrowright}(g \circ f).$$

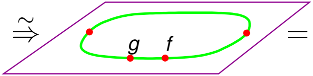
$$\mathrm{Tr}^{\circlearrowright}(f \circ g) =$$



$$\overset{\sim}{\Rrightarrow}$$



$$\overset{\sim}{\Rrightarrow}$$



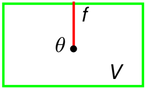
$$= \mathrm{Tr}^{\circlearrowright}(g \circ f)$$

The diagonal trace

This can be defined in a bicategory without monoidal structure.

If V is an object of a bicategory and $V \xleftarrow{f} V$ define the diagonal trace:

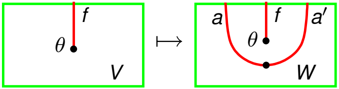
$$\mathrm{Tr}^{\searrow}(f) := 2\text{-}\mathrm{Hom}(\mathrm{Id}_V, f) = \left\{ \quad \right\}$$



Theorem (Trace property)

If $W \xleftarrow{a} V$ and $V \xleftarrow{a'} W$ with a 2-morphism $a \circ a' \overset{\eta}{\Leftarrow} Id_W$ then you get a (functorial) morphism between sets (or $\mathcal{V}$-objects):

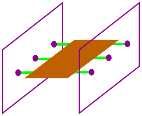
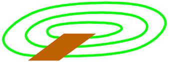
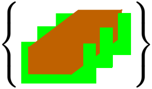
$$\mathrm{Tr}^{\searrow}(f) \xrightarrow{\eta_*} \mathrm{Tr}^{\searrow}(a \circ f \circ a')$$



In particular if $W \xleftarrow{a} V$ is an equivalence then

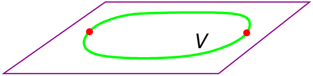
$$\mathrm{Tr}^{\searrow}(f) \cong \mathrm{Tr}^{\searrow}(a \circ f \circ a^{-1}).$$

Examples of traces in monoidal bicategories

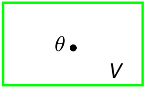
	object	endo, f	$\mathrm{Tr}^{\circlearrowright}(f)$	$\mathrm{Tr}^{\searrow}(f)$
Span	X	$\begin{tikzcd} & T \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd}$	"loops in T"	"choice of loop at each $x \in X$"
Bim	A	$_AM_A$	$M/\{ma - am\}$ coinvariants	$\{m \in M \mid am = ma\}$ invariants
$\mathcal{V}$-Mod	$\mathcal{C}$	$\mathcal{C}^{\mathrm{op}} \otimes \mathcal{C} \xrightarrow{F} \mathcal{V}$	$\int^c F(c,c)$	$\int_c F(c,c)$
2-Tang				
Var	X	$\begin{tikzcd} \mathcal{E}^\bullet \arrow[d] \\ X \times X \end{tikzcd}$	$\mathrm{HH}_\bullet(X, \mathcal{E}^\bullet)$	$\mathrm{HH}^\bullet(X, \mathcal{E}^\bullet)$
DBim	$A^\bullet$	$_{A^\bullet}M^\bullet_{A^\bullet}$	$\mathrm{HH}_\bullet(A^\bullet, M^\bullet)$	$\mathrm{HH}^\bullet(A^\bullet, M^\bullet)$

Dimension

The dimension of an object can be defined to be the trace of the identity.

$\mathrm{Dim}^{\circlearrowright}(V) := \mathrm{Tr}^{\circlearrowright}(\mathrm{Id}_V) =$ 

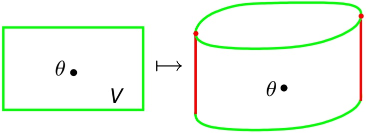
$\in 1\text{-Hom}(\mathbf{1}, \mathbf{1})$

$\mathrm{Dim}^{\searrow}(V) := \mathrm{Tr}^{\searrow}(\mathrm{Id}_V) = 2\text{-Hom}(\mathrm{Id}_V, \mathrm{Id}_V) = \Big\{$ 

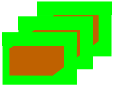
$\Big\}$

- $\mathrm{Dim}^{\searrow}(V)$ is a commutative monoid
- $\mathrm{Dim}^{\searrow}(V)$ acts on $\mathrm{Dim}^{\circlearrowright}(V)$

$$\mathrm{Dim}^{\searrow}(V) \to 2\text{-Hom}\left(\mathrm{Dim}^{\circlearrowright}(V), \mathrm{Dim}^{\circlearrowright}(V)\right)$$



Examples of dimensions in monoidal bicategories

	object, V	$\mathrm{Dim}^{\circlearrowleft}(V)$	$\mathrm{Dim}^{\searrow}(V)$
Span	X	X	$\{\star\}$
Bim	A	$A/[A, A]$	Z(A)
$\mathcal{V}$-Mod	$\mathcal{C}$	$\int^{c} \mathcal{C}(c, c)$	$\mathcal{V}$-NAT$(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}})$
2-Tang			
Var	X	$\mathrm{HH}_\bullet(X)$	$\mathrm{HH}^\bullet(X)$
DBim	$A^\bullet$	$\mathrm{HH}_\bullet(A^\bullet)$	$\mathrm{HH}^\bullet(A^\bullet)$