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The Modification Of System Transfer Functions Using Negative-Acceleration Feedback.

1. Introduction.

The analysis work described in this paper was started after completing a programme of research at Staffordshire University, which investigated the optimisation of stabilisation system performance, using rate gyroscopes to space-stabilise system load inertias.

There were four main areas to the research. One was the generation of system transfer functions, in both symbolic and numeric form. The second was the analysis of transfer functions using Mathematica software. The third was the generation of complete stabilisation system models, using Matlab / Simulink software. The fourth was the construction of a laboratory test rig, with a peak output power of 25kw. This modelled an actual control system, and could be modified to simulate other systems by changing a flywheel on the rig and the ratio of one low-power gearbox.

During the course of this work, it became obvious that one particular aspect of control theory, namely, the principle of “Standard form” expressions for transfer function denominators, could not be correct. Papers and control system textbooks discuss “Standard forms” of transfer function expressions as though it is a simple matter to change these as required by a system designer. Nothing could be further from the truth. The transfer function for any system can always be derived, in symbolic form (Often with some difficulty when the system is complicated. The mathematics is not difficult. There is just a lot of it).

- **Once the transfer function has been derived and system parameters entered, the coefficients of s in numerators and denominators calculate as being what they are and not what an analyst would like them to be.**

Over many years, on systems developed by the writer while in charge of system development work within the UK electronics industry, it was usual to employ a minor negative-acceleration feedback term to modify Nichols and Bode plots, so as to achieve the required overall system performance. This method had been in use from the early 1950’s, but the writer considered that the requirement for using this feedback had not been properly investigated, and was not really understood.

During the research at Staffordshire University, it was necessary to use this feedback method in the test rig signal-processing circuits, also in Matlab / Simulink system models, to achieve stable and accurate systems which had the required performance. On completion of the research programme at Staffordshire University, and after carrying out some further analysis on stabilisation systems, it was decided to investigate the particular feedback aspect mentioned. This produced some conclusive results concerning the writer’s disagreement with the “Standard form” concept, the correct form that these expressions take, and a set of rules governing the transfer functions of systems.

2. A Summary Of Analysis Work Carried Out On Minor Feedback Loops.

The writer has considered for some years that the “Standard form” concept for system transfer functions, given in papers and control system textbooks, is not correct. Reference to this was made previously in a paper given at a symposium held at Angers in June 2003 (AECV 2003 [1]). In September of 2003 it was decided to commence a thorough analysis of system transfer functions, with particular reference to the use of negative-acceleration feedback terms, a technique used, for example, in position and space-stabilisation systems for the Royal Navy, for the British Army, and in industrial control systems, with which the writer was involved during the design and development phases. The intention was to investigate the form of the numerators and denominators in system transfer functions, ranging from a system with one load inertia element, and with one resilience term within the system mechanics, to a system with three load inertia elements, and containing either three or four resilience terms.

The writer's objections to the accepted method of presenting such transfer functions can be explained by considering the thesis for D.Sc. by Dr. A. L. Whiteley (D.Sc. Leeds 1945 [2]), who first suggested the principle of "Standard form" expressions. The form of transfer function denominator expressions presented by Dr. Whiteley is, for example, $\left[s^4 + s^3.\omega.A + s^2.\omega^2.B + s.\omega^3.C + \omega^4\right]$ for a fourth-order-in-s denominator, and $\left[s^6 + s^5.\omega.A + s^4.\omega^2.B + s^3.\omega^3.C + s^2.\omega^4.D + s.\omega^5.E + \omega^6\right]$ for a sixth-order-in-s denominator. If this form is correct, then only one frequency would be found within the system, determined by the value of ω.

However, the oscillation frequency and damping at any particular system inertia element are determined by that particular inertia value, by its viscous friction coefficient, and by the stiffness of the resilience terms coupling it to adjacent inertias and to the driving motor. System transfer functions, and actual physical system output data, can be analysed using Mathematica software or spectrum analyser software, to show that several frequency components are present, not just a single frequency, making the examples of equation given above incorrect in form, confirmed by running Matlab / Simulink models of systems.

The mathematical analysis and modelling of a range of systems, ranging from relatively simple to extremely complex, showed that, at the motor, and at the various inertias, several frequencies will be present. The various frequency components, and their damping coefficients and phase-shifting parameters, can be identified using Mathematica software, and are also calculable. Each inertia element in the system contributes one frequency component in the transfer function denominator expression for motor speed and for the speeds of the various load inertias, which appear as identifiable frequency components after converting s-plane transfer functions to time-domain expressions.

Consequently, the form of transfer function denominator first proposed by Dr. Whiteley [2], and adopted elsewhere, cannot be correct. On page 43 of the Whiteley thesis, a list of transfer functions is presented, which are stated to produce a 10% overshoot to a step input (Equations 69 -73). For a second order system (equation 69), a 10% overshoot requires a damping coefficient of 0.595 (Figure 1). In equation 69, the damping coefficient is actually 1.25, resulting in a considerably overdamped response and not the 10% overshoot stipulated by Dr. Whiteley.

The writer has shown that the form of such transfer function denominators is not as presented, and claims to have derived the correct form for the time-domain Inverse Laplace Transforms of the system transfer functions analysed. It has also been shown that odd-order-in-s denominators are not possible, so that tables of "Standard form" equations (Whiteley thesis, page 43, and also variations of these, such as ITAE versions, published by other authors [3,4]), which contain odd-order-in-s equations, are incorrect.

The reason for this statement is quite basic. When considering any particular inertia element in a system, with inertia J_{Ln}, having a viscous friction coefficient F_{Ln}, the equation of motion at that inertia is:-
$\left[Torque\,In\;(from\;previous\;resilience\;term) - Torque\,Out\,(to\;the\;following\;resilience\;term)\right] = \left[J_{Ln}.s^2\theta_{Ln} + F_{Ln}.s\theta_{Ln}\right]$.
The result is that, as the analysis point moves through the system, from inertia to inertia, towards the motor, the order in s of the calculation increases in steps of two, as expressions are multiplied-out, so excluding odd-order-in-s denominator expressions.

On page 43 of the Whiteley thesis [2], and in several control system textbooks [3] and published papers [4], lists of standard form transfer functions, which contain odd-order equations, are presented, which the writer claims are not possible in physically realisable systems.

It has been shown that both the numerator and denominator expressions in transfer functions take the form of simple quadratics-in-s, multiplied together. At the motor, the order of the transfer function denominator depends on the number of identifiable inertia elements within the system. If N inertia elements are present, the motor speed transfer function denominator has the order $\left[\left(2\times N\right)+2\right]$ or $2\times\left[N+1\right]$, where N is the number of load inertias in the system, the remaining order of 2 coming from the denominator of the driving motor transfer function. The denominator expression in the transfer function for motor speed is also the denominator expression in the transfer functions for the speed or position of all of the inertia elements in the system (See the motor and load speed transfer functions on page 3).

The writer disagrees with Dr. Whiteley's transfer function numerator format, and has proved that the order of the motor speed transfer function numerator is always two orders in s lower than the denominator. As the point of analysis moves away from the motor to the various system inertia elements, two orders in s are lost at each step, in the numerators of transfer functions for inertia element speed or position. That is, one quadratic-in-s disappears from the speed transfer function numerator as the analysis point moves from inertia to inertia, away from the motor. The numerator quadratics only include mechanical load-inertia parameters, such as inertia, friction, and resilience stiffness coefficients, and do not contain any motor parameters.

The paper next presents the conclusions reached by the analysis as a list of rules and discusses how these rules were verified.

3. List Of Rules Governing The Order Of System Transfer Function Numerators And Denominators.

In order to illustrate the mathematical form of system transfer functions, to which the following rules apply, consider first of all one of the more complex systems analysed, comprising three identifiable inertia elements, coupled to each other and to the driving motor by three resilience terms, with different viscous friction coefficients at each of the inertias. It has been proved that the form of the motor-speed $s\theta_m$ transfer function and of the three load inertia speed transfer functions $s\theta_{L1},\ s\theta_{L2},\ and\ s\theta_{L3}$ consist of quadratics in s, and are of the form given below.

$$s\theta_m = K_1 \cdot \frac{\left[\left\{s^2 + s.2.\xi_5.\omega_5 + \omega_5^{\,2}\right\} \times \left\{s^2 + s.2.\xi_6.\omega_6 + \omega_6^{\,2}\right\} \times \left\{s^2 + s.2.\xi_7.\omega_7 + \omega_7^{\,2}\right\}\right]}{\left[\left\{s^2 + s.2.\xi_1.\omega_1 + \omega_1^{\,2}\right\} \times \left\{s^2 + s.2.\xi_2.\omega_2 + \omega_2^{\,2}\right\} \times \left\{s^2 + s.2.\xi_3.\omega_3 + \omega_3^{\,2}\right\} \times \left\{s^2 + s.2.\xi_4.\omega_4 + \omega_4^{\,2}\right\}\right]}$$

$$s\theta_{L1} = K_2 \cdot \frac{\left[\left\{s^2 + s.2.\xi_8.\omega_8 + \omega_8^{\,2}\right\} \times \left\{s^2 + s.2.\xi_9.\omega_9 + \omega_9^{\,2}\right\}\right]}{\left[\left\{s^2 + s.2.\xi_1.\omega_1 + \omega_1^{\,2}\right\} \times \left\{s^2 + s.2.\xi_2.\omega_2 + \omega_2^{\,2}\right\} \times \left\{s^2 + s.2.\xi_3.\omega_3 + \omega_3^{\,2}\right\} \times \left\{s^2 + s.2.\xi_4.\omega_4 + \omega_4^{\,2}\right\}\right]}$$

$$s\theta_{L2} = K_3 \cdot \frac{\left[\left\{s^2 + s.2.\xi_{10}.\omega_{10} + \omega_{10}^{\,2}\right\}\right]}{\left[\left\{s^2 + s.2.\xi_1.\omega_1 + \omega_1^{\,2}\right\} \times \left\{s^2 + s.2.\xi_2.\omega_2 + \omega_2^{\,2}\right\} \times \left\{s^2 + s.2.\xi_3.\omega_3 + \omega_3^{\,2}\right\} \times \left\{s^2 + s.2.\xi_4.\omega_4 + \omega_4^{\,2}\right\}\right]}$$

$$s\theta_{L3} = K_4 \cdot \frac{1}{\left[\left\{s^2 + s.2.\xi_1.\omega_1 + \omega_1^{\,2}\right\} \times \left\{s^2 + s.2.\xi_2.\omega_2 + \omega_2^{\,2}\right\} \times \left\{s^2 + s.2.\xi_3.\omega_3 + \omega_3^{\,2}\right\} \times \left\{s^2 + s.2.\xi_4.\omega_4 + \omega_4^{\,2}\right\}\right]}$$

The constants K_1, K_2, K_3, and K_4 are speed-scaling constants. For example, in the motor-speed transfer function, $K_1 = \left[\frac{Input\ Voltage \times Motor\ Controller\ Voltage\ Gain \times Motor\ Torque\ Constant}{Motor\ Inertia \times Motor\ Inductance}\right]$

The following rules have been shown to apply.

1. At the motor, the order-in-s of the motor speed transfer function denominator is $2.\left[N+1\right]$, where there are N inertia elements present in the system.

2. The motor speed transfer function numerator is always two orders in s less than the denominator.

3. As the analysis point moves away from the motor to load inertia J_{L1}, then to J_{L2} and to J_{L3}, one numerator quadratic is lost at each step, but note that the numerator quadratics in the expressions for $s\theta_{L1}$, $s\theta_{L2}$, and $s\theta_{L3}$ all contain different quadratic expressions in s, which have different damping and frequency components (The four equations printed above attempt to illustrate this).

4. The order-in-s of the denominator expression is determined by the number of inertia terms present (Rule 1). The denominator expression also remains constant throughout. It is the common

denominator expression in all of the motor and load-inertia transfer functions for the particular system being considered, and it is also the denominator in the transfer function for motor current.

5. If an extra inertia is introduced, but if this is not decoupled by an additional resilience element, the order of the motor speed denominator remains unchanged. That is, each inertia element must be decoupled from its neighbour by a resilience term, otherwise the order of both the numerator and denominator expressions does not change. Common-sense reasoning states that this will be the case, since, inserting an extra undecoupled inertia produces a system in which the new inertia and its neighbour can be lumped in the mathematics as a single inertia, hence the system order is unchanged, the added inertia having no independent freedom of movement.

6. Transfer function numerator expressions contain only parameters associated with the system mechanics, and do not contain any motor parameters.

7. The numerator expression in the transfer function for motor speed, or in the transfer function for any one of the system load inertias, is the "Locked rotor" equation for the next inertia in the system, moving away from the motor, i.e., the numerator in the motor speed transfer function is the locked rotor equation for the next inertia in the system, one removed from the motor. The numerator in the speed transfer function for that particular inertia is the locked rotor equation for the second inertia from the motor, etc.

8. When transfer functions for motor speed or for load-inertia speed are converted to the time domain, using the Mathematica InverseLaplaceTransform command, for example, the number of frequency components in the time functions generated depends on the number of quadratic expressions in the transfer function denominator, in turn determined by the number of inertias in the system. Each denominator quadratic contributes one frequency component to time-domain responses.

9. Numerator quadratics do not provide frequency components, but only contribute to exponential damping and phase-shifting / scaling terms in time-domain expressions for motor speed or for load inertia speed.

These rules apply to any mechanical drive arrangement, driven by a motor. The analysis work was eventually extended to a system with three inertias, and which included not three but four resilience terms. The transfer function for motor speed in the latter case contains 33 terms in the numerator sixth-order-in-s expression, and 225 terms in the eighth-order-in-s denominator. Since an extra resilience was introduced but no additional inertia, the motor speed and load-inertia speed transfer function numerator and denominator brackets remained at the same order in s, as for the 3-inertia / 3-resilience case, although the mathematics of the analysis became more complex than for the three-resilience system and the resulting equations were also more complex.

4. The Use Of A Negative-Acceleration Minor Feedback Loop To Modify Equation Coefficients And To Adjust The System Motor And Load Inertia Responses.

The minor loop feedback method referred to, used for many years, employed a "Transient motor voltage" feedback term, generated by picking off motor volts at the motor terminals, processing this via a signal-processing lead / lag network, scaling the resulting output voltage, and using this as the minor loop feedback signal. The resulting feedback signal contains several components, since the voltage at the motor terminals must at all times equal the voltage applied to the motor by the motor controller output voltage V_0, and $V_0 = K_V.s\theta_m + I_a.\left[R_a + s.L_a\right]$. After processing this voltage via a resistor/capacitor signal-processing network, the effect of each term becomes somewhat obscure (The signal-processing lag / lead frequencies are adjustable, and the feedback signal-processing transfer function itself is quite complex). It was therefore decided to carry out the analysis using what might be described as "pure" terms, using motor instantaneous acceleration (Actual motor acceleration), or, alternatively, motor instantaneous torque ("Possible" motor acceleration), in order to identify the effect of each parameter on the overall system response.

There has never been any publication concerning methods of achieving the desired "Standard form" coefficients-of-s, i.e. the design of suitable hardware to achieve this aim. This paper claims that the only calculable and mathematically justifiable method of achieving the desired result, namely, the adjustment of particular coefficients-of-s in system transfer functions, is to use a variable negative-acceleration feedback term around the servo motor and its motor controller unit. In actual systems this is used in conjunction with outer-loop (A position or rate pickoff loop) signal processing and error-channel (Forward sequence) signal processing. The minor loop feedback does not solve all of the control system problems entirely on its own.

Two of the minor loop feedback methods analysed were

1. Use a good-quality permanent-magnet tachometer unit, coupled very rigidly to the shaft of the servo motor. The output of the tacho represents actual motor speed. If this signal is differentiated and scaled by the feedback system, the resulting feedback signal represents motor instantaneous acceleration, the feedback voltage being $A_2 \times K_{TACHO} \times \left(s.Motor\ Speed \right)$, or $A_2 \times K_{TACHO} \times s^2\theta_m$ where the tacho constant is in volts/radian/second, and A_2 is the gain of the signal-scaling feedback amplifier.

2. Monitor motor current using a current shunt, again scaling the feedback voltage generated and feeding this back to the input of the motor controller unit, the feedback voltage being $A_2 \times R_S \times Motor\ Current$. R_S is the shunt resistance and A_2 is the gain of the feedback scaling amplifier.

(5)

The possibility of using negative acceleration feedback derived from a load tacho has also been fully-analysed, but is not presented in this paper, the changes to system performance looked for being considerably less than when applying either of the two methods described above.

Using a motor tacho would require such a device to be fitted to the shaft of every servo motor. This is often done in U.K. Military and Naval systems, to allow an unstabilised (But still closed-loop) fallback mode of operation in the event of a gyroscope failure. In industrial control systems, the added requirement for a motor tacho to be fitted, plus a differentiator circuit and feedback scaling amplifier, would be a total nuisance, but this method at least puts forward one possible feedback method for consideration in the mathematics. The second method is clearly to be preferred. Motor controller units have a low-inductance resistive current shunt fitted, or a Hall-effect pickoff, to monitor motor current. With permanent-magnet servo motors, the motor current should not exceed the motor manufacturer’s stated peak figure, otherwise demagnetisation of the motor magnetic circuit will occur, and so the same current pickoff could be used to generate a negative-acceleration feedback signal.

The feedback signal from the current shunt represents instantaneous motor torque. Differentiated motor tacho feedback represents instantaneous motor acceleration. During the analysis and modelling work, it was shown that the effect of these two alternative feedback methods is quite different. Motor tacho feedback suppresses the intrinsic motor startup oscillation precisely, on a calculable basis, but does not significantly improve the load response. Motor current feedback damps load oscillations particularly and also improves the motor response. Additionally, it considerably reduces stress levels within the system mechanics.

The modelling and analysis both showed that the amount of motor current feedback needed to produce critical damping of an unloaded servo motor, at startup, must be significantly reduced when the motor drives a complex mechanical system containing resilience terms. On applying current feedback at the level calculated to critically damp an unloaded motor, the load response is then grossly overdamped, and the load transient response is too slow.

5. The Suppression Of Motor Startup Oscillations.

This section of the paper considers the startup of an unloaded servo motor. The reader will ask why an unloaded motor is being considered, since the function of a motor is to drive things. The different effects of motor tacho feedback and motor current feedback are introduced here. It will be shown that tacho feedback can be used to critically-damp an unloaded motor at startup, without affecting the transient oscillation frequency of the motor, whereas current feedback will also produce critical motor damping, but at the same time, the motor transient oscillating frequency is changed. This difference applies similarly to loaded motors driving one inertia via one resilience element, to motors driving three inertias via three resilience terms.

In order to save time, computer models for the minor feedback loop investigation, now under discussion, were derived from larger models of complete stabilisation systems, developed during earlier research, by simply removing the outer rate-gyro feedback loop and all its associated Simulink blocks, together with the carrier-vehicle hull-disturbance circuitry. For reasons of consistency, a gain block, such as the motor controller voltage gain in the stabilisation system, retained its original designation K_A throughout the minor loop analysis, and the scaling amplifier feeding back either a motor tacho or a motor current signal was still A_2 in the analysis of the minor feedback loop. All parameters are defined on page 16 of the paper.

The transfer function for an unloaded servo motor is

$$s\theta_m = \frac{K_A.K_T.V_{IN}}{J_m.L_a}.\frac{1}{\left[s^2 + s.\left\{\frac{R_a}{L_a} + \frac{F_m}{J_m}\right\} + \left\{\frac{K_V.K_T}{J_m.L_a} + \frac{F_m.R_a}{J_m.L_a}\right\}\right]}$$

. The form of the denominator bracket is $\left[s^2 + s.2.\xi.\omega + \omega^2\right]$. If values are applied as for the S.E.M. brushless motor used during research at Staffordshire University, then $s\theta_m = 17857143.V_{in}.\frac{1}{\left[s^2 + s.72.93 + 134035.7\right]}$.

The motor damping $\xi = 0.0966$ and $\omega = 366.11$, so that motor startup is oscillatory.

Now couple a tacho to the motor with an output constant of K_{TACHO} volts per radian per second, and feed this signal back to the input of the motor controller via a scaling amplifier with a gain of A_2, plus a unity differentiator. The feedback signal is $A_2.K_{TACHO}.s^2\theta_m$.

(6)

The transfer function for motor speed becomes

$$s\theta_m = \frac{K_A.K_T.V_{IN}}{J_m.L_a}.\frac{1}{\left[s^2 + s.\left\{\frac{R_a}{L_a} + \frac{F_m}{J_m} + \frac{K_A.K_T.A_2.K_{TACHO}}{J_m.L_a}\right\} + \left\{\frac{K_V.K_T}{J_m.L_a} + \frac{F_m.R_a}{J_m.L_a}\right\}\right]}$$

. It is seen that the level of tacho feedback needed to produce critical motor damping is easily calculable, since the feedback has not modified the "ω^2" denominator bracket $\left\{\frac{K_V.K_T}{J_m.L_a} + \frac{F_M.R_a}{J_m.L_a}\right\}$.

The motor damping is now $\xi = \left[\frac{\frac{F_m}{J_m} + \frac{R_a}{L_a} + \frac{K_A.K_T.A_2.K_{TACHO}}{J_m.L_a}}{2.\sqrt{\frac{F_m.R_a}{J_m.L_a} + \frac{K_V.K_T}{J_m.L_a}}}\right]$ and if ξ *must equal* 1.0, then

$A_2.K_{TACHO} = 3.69203.10^{-5}$ and $s\theta_m = 17857143.V_{IN}.\frac{1}{\left[s^2 + s.732.22 + 134035.7\right]}$. The damping ξ is now increased to 1.0, as required, but ω is still 366.11.

Alternatively, use a motor current shunt, value R_S ohms. The feedback signal generated is $R_S.I_a$. After scaling this signal by amplifier A_2, the feedback voltage is $A_2.R_S.I_a$, and the motor speed transfer function becomes

$$s\theta_m = \frac{K_A.K_T.V_{IN}}{J_m.L_a}.\frac{1}{\left[s^2 + s.\left\{\frac{F_m}{J_m} + \frac{R_a}{L_a} + \frac{K_A.A_2.R_S}{L_a}\right\} + \left\{\frac{K_V.K_T}{J_m.L_a} + \frac{F_m.R_a}{J_m.L_a} + \frac{F_m.A_2.R_S.K_A}{J_m.L_a}\right\}\right]}$$

In this case the natural frequency of the motor has been modified, as well as the motor damping, making it rather more difficult to calculate the precise level of current feedback needed for any particular value of motor damping.

The equation for the damping coefficient is now $\xi = \frac{\left[\frac{F_m}{J_m} + \frac{R_a}{L_a} + \frac{K_A.A_2.R_S}{L_a}\right]}{2.\sqrt{\frac{K_V.K_T}{J_m.L_a} + \frac{F_m.R_a}{J_m.L_a} + \frac{F_m.A_2.R_S.K_A}{J_m.L_a}}}$.

To achieve a motor damping factor of 1.0, the exact value of $A_2.R_S = 0.013908$. In order to obtain an approximate solution quickly, assume that there has not been any change in motor ω_n due to the current feedback term, when $A_2.R_S$ calculates as being $A_2.R_S = 0.0138451$, a difference of only 0.452% from the exact calculated feedback level, since the term $\frac{F_m.A_2.R_S.K_A}{J_m.L_a} \ll \left\{\frac{K_V.K_T}{J_m.L_a} + \frac{F_m.R_a}{J_m.L_a}\right\}$ $(993 \ll 134035.7)$

Figure 2 is a Matlab / Simulink model of an unloaded motor, which allows motor speed, motor current, motor copper loss, etc, to be plotted. All parameters can be changed in the model. Figure 3 plots motor speed without feedback (Red plot), with tacho feedback applied (Green plot), and with current feedback applied (Blue plot), the feedback being applied at the levels calculated above.

A second way of producing the same motor speed plots is to use a much more basic Matlab / Simulink model, illustrated by Figure 5, in which the three unloaded motor speed transfer functions are entered as single numeric transfer functions for the three cases discussed (No feedback, tacho feedback, and current feedback), but this model can clearly only be used to represent other types of motor by re-calculating and changing all three transfer functions.

A third option is to derive the numeric transfer function Inverse Laplace Transforms using Mathematica, and then instruct Mathematica to plot the time responses. Time responses for the startup of an unloaded motor without any feedback applied, also with tacho and motor current feedback applied, were obtained using all three methods, for comparison, and motor speed plots using the three methods were identical. Figure 3 shows a combined plot for motor speed without any feedback, also with the correct levels of tacho or current feedback applied. The effect on the motor damping at startup is obvious.

Figure 4 plots the same response to 30ms. There is very little difference between the plots using tacho and current feedback, but the slight increase in final motor speed error using current feedback can be made out, and can be measured accurately by using the Matlab “axis” command to look more closely at the time period 25ms to 30ms and the speed range 148 to 152 rads/sec.

The analysis has so far illustrated the principle of applying tacho or current feedback to an unloaded motor. Tacho feedback does not modify the denominator “ω^2” term, whereas current feedback increases the “ω^2” term as well an changing the motor damping. This applies similarly to transfer functions for more complex systems with one, two, and three inertia elements, when the final denominator term has the dimensions ω^4, ω^6, or ω^8.

Now connect the motor to a load inertia via a reduction gearbox, reduction ratio N:1, but consider that the gearbox is infinitely stiff, i.e., that it does not deflect under torsion (It has an infinite resilience stiffness figure). If the load inertia is J_L and the load viscous friction figure is F_L, from basic control theory, the total effective inertia at the motor is now $\left[J_m + \frac{J_L}{N^2}\right]$ and the friction at the motor is $\left[F_m + \frac{F_L}{N^2}\right]$.

If J_L = 35,000 kg.m^2 and if F_L = 60 Nm/deg./sec. (3437.8 Nm/radian/sec. load speed), then, for N = 800, the motor inertia becomes 0.056688 and the motor friction becomes 0.008372 (F_m = 0.003, J_m = 0.002).

The motor ω_n becomes 68.816 and the damping is 0.5201. The lightly-damped high-frequency motor startup oscillation has been modified to a low-frequency and well-damped oscillation. Figure 6 plots motor startup for such a loaded motor driving a 35,000 kg.m^2 load inertia via an 800:1 gear reduction, but there is no resilience term within the mechanics. Unfortunately for system designers, this does not represent a “real world” situation, otherwise control system designers would not have any problems. No mechanical drive has infinite torsional rigidity. The method of referring load inertia and friction to the motor in this fashion is merely academic, not practical. The correct form for the loaded motor situation (“Real world” model) should be a motor loaded by a load inertia, but with a resilience element placed at some point within the system.

In an actual control system, modelled by the test rig at Staffordshire University, the drive comprises a 3-stage planetary reduction gearbox driven by the servo motor, reduction ratio 46.24, whose output pinion engages a 2-metre diameter gear, which turns the load inertia. The second gear ratio is 17.3 (Gearbox output pinion to 2-metre gear reduction ratio), so that the overall gear reduction, motor to load inertia, is almost precisely 800:1.

The resilience term in the actual system comprises windup of the first reduction gearbox, some deflection of the gearbox mounting bracket, and some deflection of the second gear reduction, so that the whole can be lumped as an equivalent resilience term K_S situated between the first gear reduction and the second stage of gear reduction. The two gear reductions can now be regarded as perfect reduction gearbox assemblies, with reduction ratios N_1 and N_2, each having infinite torsional stiffness. On the test rig at Staffordshire University, a 1-metre steel bar provided the resilience term.

6. A Single Inertia System, With A Single Resilience Element Situated Within The Geartrain.

The introduction of a resilience element immediately allows the re-appearance of the motor high-frequency oscillation. The transfer functions for the motor speed and load speed for this case are presented on pages 9, 10, and 11 of the paper, in landscape format for ease of reading. If negative acceleration feedback is applied, only the denominator expression changes. Additional terms are added to the various coefficients of s in the denominator only.

Using tacho feedback, the additional terms are, $s^3 = \frac{K_A.K_T.A_2.K_{TACHO}}{J_m.L_a}$, $s^2 = \frac{K_A.K_T.A_2.K_{TACHO}.F_L}{J_L.J_m.L_a}$,

$s = \frac{K_A.K_T.A_2.K_{TACHO}.K_S.N_2^{\,2}}{J_L.J_m.L_a}$ There is no addition to the last denominator term, whose dimensions are "ω^4", when using tacho feedback. (Page 10 equations).

Using current feedback, additional terms are, $s^3 = \left\{\frac{K_A.A_2.R_S}{L_a}\right\}$, $s^2 = \left\{\frac{F_m.K_A.A_2.R_S}{J_m.L_a} + \frac{F_L.K_A.A_2.R_S}{J_L.L_a}\right\}$,

$s = \left\{\frac{F_m.K_A.A_2.R_S.F_L}{J_L.J_m.L_a} + \frac{K_A.A_2.R_S.K_S.N_2^{\,2}}{J_L.L_a} + \frac{K_A.A_2.R_S.K_S}{J_m.L_a.N_1^{\,2}}\right\}$, "$\omega^4$" = $\left\{\frac{K_A.A_2.R_S.K_S.F_L}{N_1^{\,2}.J_L.J_m.L_a} + \frac{K_A.A_2.R_S.K_S.F_m.N_2^{\,2}}{J_L.J_m.L_a}\right\}$

(Page 11 equations).

(8)

The final denominator bracket has been modified in the latter case by the addition of two extra terms. The use of current feedback consequently increases the system speed error, similar in effect to increasing system friction. Using motor current feedback, when motor acceleration transients have died out, some motor current is still required, to drive both the motor and load friction terms, producing a negative feedback signal from the current shunt. This subtracts from the incoming speed demand voltage to the motor controller, reducing the steady-state motor speed and marginally increasing the steady-state speed error.

Figure 7 shows a Matlab/Simulink model, which models a motor driving a load inertia via two gear reductions, with a resilience positioned between the gear sections. All of the writer's models developed during the research have the facility to add an additional "Speed-squared" term (Gain 21) to the load viscous friction (Gain 4), also static friction (Constant 14). It was found that the 2-stage planetary gearbox attached to the brushless motor on the servo test rig at Staffordshire University, had an extremely non-linear torque/speed characteristic. An extra speed-squared term in Matlab / Simulink models allows the gearbox non-linear torque/speed characteristic to be modelled correctly.

Figure 8 is a plot showing startup of the loaded motor without any feedback applied (Red plot. The high-frequency unloaded-motor oscillation has now re-appeared), using tacho feedback (Green plot), and using current feedback (Blue plot). Figure 9 plots the load speed responses similarly. From these two sets of graphs it can be seen that, using the correct level of current feedback for optimum load response, the motor damping is less than unity (the motor oscillation frequency is still present, but with lower amplitude and greater damping than when operating without any feedback applied), but the current feedback has critically-damped the load response.

The model (Figure 7) allows the windup angle and torque within the resilience element to be plotted. Figure 10 shows that, using current feedback, a reduced stress level within the system mechanics results. The motor current graphs for the three cases (Figure 11) show that, without feedback, or using tacho feedback, the motor current peaks at about the setting of the motor current clamping circuit (300 amps), for an input voltage step of 1.125 volts. Using current feedback, the peak motor current reduces to 185 amps. Motor heating, due to the lowering of the motor I^2.R copper loss, also reduces appreciably, Figure 12.

Figure 13 shows that the peak motor power required is significantly reduced when current feedback is used (27.5 kw reduces to 8 kw using current feedback), which also means that the peak power required from the motor controller reduces from 34 kw to 10 kw (Figure 14).

It will now be explained how the correct form for the motor and load speed transfer functions was verified. It was firstly attempted to analyse the denominator brackets of fourth, sixth, and eighth order-in-s transfer function denominators, and also their numerator expressions, using symbolic methods. If the form of the denominator is (say) $\left[s^6 + s^5.A + s^4.B + s^3.C + s^2.D + s.E + F\right]$, using the "Roots" command in Mathematica will generate some 1000 pages of A4 solution, which is correct mathematics but is not understandable. The solution to this problem was to convert from symbolic to numeric form by putting system values into the transfer function numerator and denominators for motor and load speed. Using the numeric versions, Mathematica then produced sets of roots that were easily understandable. These comprised pairs of conjugate complex roots, which, multiplied-out as pairs, generate simple quadratics in s, making the frequency and damping components obvious in both the numerator and denominator expressions.

The next three pages present motor and load speed transfer functions for a motor, driving a load inertia via reduction gearing, with a single resilience element within the gearing, in landscape format. The three cases are

1. No feedback applied.
2. Tacho feedback applied.
3. Current feedback applied.

(The text resumes on page 12).

(9)

The transfer function calculated for the motor speed for the case of a single inertia being driven by a servo motor, with a single resilience element K_S within the reduction gearing, located between the first gearing reduction N_1 and the second gearing reduction N_2, and <u>without any feedback applied</u>, is

$$s\theta_m = \frac{V_{IN}.K_A.K_T}{J_m.L_a} \times \frac{\left[s^2 + s.\frac{F_L}{J_L} + \frac{K_S.N_2^2}{J_L} \right]}{\left[s^4 + s^3.\left\{ \frac{R_a}{L_a} + \frac{F_m}{J_m} + \frac{F_L}{J_L} \right\} + s^2.\left\{ \begin{array}{l} \frac{K_V.K_T}{J_m.L_a} + \frac{F_m.R_a}{J_m.L_a} + \frac{K_S}{N_1^2.J_m} + \\ \frac{F_L.R_a}{J_L.L_a} + \frac{F_m.F_L}{J_m.J_L} + \frac{K_S.N_2^2}{J_L} \end{array} \right\} + s.\left\{ \begin{array}{l} \frac{K_S.R_a}{N_1^2.J_m.L_a} + \frac{K_V.K_T.F_L}{J_m.J_L.L_a} + \frac{F_m.F_L.R_a}{J_m.J_L.L_a} + \\ \frac{K_S.F_L}{N_1^2.J_L.J_m} + \frac{K_S.R_a.N_2^2}{J_L.L_a} + \frac{F_m.K_S.N_2^2}{J_L.J_m} \end{array} \right\} + \left\{ \begin{array}{l} \frac{K_S.R_a.F_L}{N_1^2.J_m.J_L.L_a} + \frac{F_m.R_a.K_S.N_2^2}{J_L.J_m.L_a} + \\ \frac{K_V.K_T.K_S.N_2^2}{J_L.J_m.L_a} \end{array} \right\} \right]}$$

The transfer function for the load speed for this case is

$$s\theta_L = \frac{V_{IN}.K_A.K_T.K_S.N_2}{J_m.J_L.N_1.L_a} \times \frac{1}{\left[s^4 + s^3.\left\{ \frac{R_a}{L_a} + \frac{F_m}{J_m} + \frac{F_L}{J_L} \right\} + s^2.\left\{ \begin{array}{l} \frac{K_V.K_T}{J_m.L_a} + \frac{F_m.R_a}{J_m.L_a} + \frac{K_S}{N_1^2.J_m} + \\ \frac{F_L.R_a}{J_L.L_a} + \frac{F_m.F_L}{J_m.J_L} + \frac{K_S.N_2^2}{J_L} \end{array} \right\} + s.\left\{ \begin{array}{l} \frac{K_S.R_a}{N_1^2.J_m.L_a} + \frac{K_V.K_T.F_L}{J_m.J_L.L_a} + \frac{F_m.F_L.R_a}{J_m.J_L.L_a} + \\ \frac{K_S.F_L}{N_1^2.J_L.J_m} + \frac{K_S.R_a.N_2^2}{J_L.L_a} + \frac{F_m.K_S.N_2^2}{J_L.J_m} \end{array} \right\} + \left\{ \begin{array}{l} \frac{K_S.R_a.F_L}{N_1^2.J_m.J_L.L_a} + \frac{F_m.R_a.K_S.N_2^2}{J_L.J_m.L_a} + \\ \frac{K_V.K_T.K_S.N_2^2}{J_L.J_m.L_a} \end{array} \right\} \right]}$$

(10)

Using motor tacho feedback, the transfer functions for motor speed and load speed become

$$s\theta_m = \frac{V_{IN}.K_A.K_T}{J_m.L_a} \times \frac{\left[s^2 + s.\frac{F_L}{J_L} + \frac{K_S.N_2^{\,2}}{J_L} \right]}{\left[s^4 + s^3.\left\{ \begin{array}{l} \frac{R_a}{L_a} + \frac{F_m}{J_m} + \frac{F_L}{J_L} \\ + \frac{K_A.K_T.A_2.K_{TACHO}}{J_m.L_a} \end{array} \right\} + s^2.\left\{ \begin{array}{l} \frac{K_V.K_T}{J_m.L_a} + \frac{F_m.R_a}{J_m.L_a} + \frac{K_S}{N_1^{\,2}.J_m} \\ + \frac{F_L.R_a}{J_L.L_a} + \frac{F_m.F_L}{J_m.J_L} + \frac{K_S.N_2^{\,2}}{J_L} \\ + \frac{K_A.K_T.A_2.K_{TACHO}.F_L}{J_L.J_m.L_a} \end{array} \right\} + s.\left\{ \begin{array}{l} \frac{K_S.R_a}{N_1^{\,2}.J_m.L_a} + \frac{K_V.K_T.F_L}{J_m.J_L.L_a} + \frac{F_m.F_L.R_a}{J_m.J_L.L_a} + \\ \frac{K_S.F_L}{N_1^{\,2}.J_L.J_m} + \frac{K_S.R_a.N_2^{\,2}}{J_L.L_a} + \frac{F_m.K_S.N_2^{\,2}}{J_L.J_m} \\ + \frac{K_A.K_T.A_2.K_{TACHO}.K_S.N_2^{\,2}}{J_L.J_m.L_a} \end{array} \right\} + \left\{ \begin{array}{l} \frac{K_S.R_a.F_L}{N_1^{\,2}.J_m.J_L.L_a} + \frac{F_m.R_a.K_S.N_2^{\,2}}{J_L.J_m.L_a} + \\ \frac{K_V.K_T.K_S.N_2^{\,2}}{J_L.J_m.L_a} \end{array} \right\} \right]}$$

The transfer function for the load speed for this case is

$$s\theta_L = \frac{V_{IN}.K_A.K_T.K_S.N_2}{J_m.J_L.N_1.L_a} \times \frac{1}{\left[s^4 + s^3.\left\{ \begin{array}{l} \frac{R_a}{L_a} + \frac{F_m}{J_m} + \frac{F_L}{J_L} \\ + \frac{K_A.K_T.A_2.K_{TACHO}}{J_m.L_a} \end{array} \right\} + s^2.\left\{ \begin{array}{l} \frac{K_V.K_T}{J_m.L_a} + \frac{F_m.R_a}{J_m.L_a} + \frac{K_S}{N_1^{\,2}.J_m} \\ + \frac{F_L.R_a}{J_L.L_a} + \frac{F_m.F_L}{J_m.J_L} + \frac{K_S.N_2^{\,2}}{J_L} \\ + \frac{K_A.K_T.A_2.K_{TACHO}.F_L}{J_L.J_m.L_a} \end{array} \right\} + s.\left\{ \begin{array}{l} \frac{K_S.R_a}{N_1^{\,2}.J_m.L_a} + \frac{K_V.K_T.F_L}{J_m.J_L.L_a} + \frac{F_m.F_L.R_a}{J_m.J_L.L_a} \\ + \frac{K_S.F_L}{N_1^{\,2}.J_L.J_m} + \frac{K_S.R_a.N_2^{\,2}}{J_L.L_a} + \frac{F_m.K_S.N_2^{\,2}}{J_L.J_m} \\ + \frac{K_A.K_T.A_2.K_{TACHO}.K_S.N_2^{\,2}}{J_L.J_m.L_a} \end{array} \right\} + \left\{ \begin{array}{l} \frac{K_S.R_a.F_L}{N_1^{\,2}.J_m.J_L.L_a} + \frac{F_m.R_a.K_S.N_2^{\,2}}{J_L.J_m.L_a} + \\ \frac{K_V.K_T.K_S.N_2^{\,2}}{J_L.J_m.L_a} \end{array} \right\} \right]}$$

Using motor tacho feedback, one term has been added to the coefficient of s^3, one term has been added to the coefficient of s^2, one term has been added to the coefficient of s, but the coefficient of “ω^4” has not been modified, in either of the transfer function denominators.

(11)

Using <u>motor current feedback</u>, the transfer functions for motor speed and load speed become

$$s\theta_m = \frac{V_{IN}.K_A.K_T}{J_m.L_a} \times \frac{\left[s^2 + s.\frac{F_L}{J_L} + \frac{K_S.N_2^{\ 2}}{J_L} \right]}{\left[\begin{array}{l} s^4 + s^3.\left\{ \begin{array}{l} \frac{R_a}{L_a} + \frac{F_m}{J_m} + \frac{F_L}{J_L} \\ + \frac{K_A.A_2.R_S}{L_a} \end{array} \right\} + s^2.\left\{ \begin{array}{l} \frac{K_V.K_T}{J_m.L_a} + \frac{F_m.R_a}{J_m.L_a} + \frac{K_S}{N_1^{\ 2}.J_m} \\ + \frac{F_L.R_a}{J_L.L_a} + \frac{F_m.F_L}{J_m.J_L} + \frac{K_S.N_2^{\ 2}}{J_L} \\ + \frac{F_m.K_A.A_2.R_S}{J_m.L_a} + \frac{F_L.K_A.A_2.R_S}{J_L.L_a} \end{array} \right\} + s.\left\{ \begin{array}{l} \frac{K_S.R_a}{N_1^{\ 2}.J_m.L_a} + \frac{K_V.K_T.F_L}{J_m.J_L.L_a} + \frac{F_m.F_L.R_a}{J_m.J_L.L_a} \\ + \frac{K_S.F_L}{N_1^{\ 2}.J_L.J_m} + \frac{K_S.R_a.N_2^{\ 2}}{J_L.L_a} + \frac{F_m.K_S.N_2^{\ 2}}{J_L.J_m} \\ + \frac{F_m K_A.A_2.R_S.F_L}{J_L.J_m.L_a} + \frac{K_A.A_2.R_S.K_S.N_2^{\ 2}}{J_L.L_a} \\ + \frac{K_A.A_2.R_S.K_S}{J_m.L_a.N_1^{\ 2}} \end{array} \right\} + \left\{ \begin{array}{l} \frac{K_S.R_a.F_L}{N_1^{\ 2}.J_m.J_L.L_a} + \frac{F_m.R_a.K_S.N_2^{\ 2}}{J_L.J_m.L_a} + \\ \frac{K_V.K_T.K_S.N_2^{\ 2}}{J_L.J_m.L_a} + \frac{K_A.A_2.R_S.K_S.F_L}{N_1^{\ 2}.J_L.J_m.L_a} + \\ \frac{K_A.A_2.R_S.K_S.F_m.N_2^{\ 2}}{J_L.J_m.L_a} \end{array} \right\} \end{array} \right]}$$

The transfer function for the load speed for this case is

$$s\theta_L = \frac{V_{IN}.K_A.K_T.K_S.N_2}{J_m.J_L.N_1.L_a} \times \frac{1}{\left[\begin{array}{l} s^4 + s^3.\left\{ \begin{array}{l} \frac{R_a}{L_a} + \frac{F_m}{J_m} + \frac{F_L}{J_L} \\ + \frac{K_A.A_2.R_S}{L_a} \end{array} \right\} + s^2.\left\{ \begin{array}{l} \frac{K_V.K_T}{J_m.L_a} + \frac{F_m.R_a}{J_m.L_a} + \frac{K_S}{N_1^{\ 2}.J_m} \\ + \frac{F_L.R_a}{J_L.L_a} + \frac{F_m.F_L}{J_m.J_L} + \frac{K_S.N_2^{\ 2}}{J_L} \\ + \frac{F_m.K_A.A_2.R_S}{J_m.L_a} + \frac{F_L.K_A.A_2.R_S}{J_L.L_a} \end{array} \right\} + s.\left\{ \begin{array}{l} \frac{K_S.R_a}{N_1^{\ 2}.J_m.L_a} + \frac{K_V.K_T.F_L}{J_m.J_L.L_a} + \frac{F_m.F_L.R_a}{J_m.J_L.L_a} \\ + \frac{K_S.F_L}{N_1^{\ 2}.J_L.J_m} + \frac{K_S.R_a.N_2^{\ 2}}{J_L.L_a} + \frac{F_m.K_S.N_2^{\ 2}}{J_L.J_m} \\ + \frac{F_m K_A.A_2.R_S.F_L}{J_L.J_m.L_a} + \frac{K_A.A_2.R_S.K_S.N_2^{\ 2}}{J_L.L_a} \\ + \frac{K_A.A_2.R_S.K_S}{J_m.L_a.N_1^{\ 2}} \end{array} \right\} + \left\{ \begin{array}{l} \frac{K_S.R_a.F_L}{N_1^{\ 2}.J_m.J_L.L_a} + \frac{F_m.R_a.K_S.N_2^{\ 2}}{J_L.J_m.L_a} + \\ \frac{K_V.K_T.K_S.N_2^{\ 2}}{J_L.J_m.L_a} + \frac{K_A.A_2.R_S.K_S.F_L}{N_1^{\ 2}.J_L.J_m.L_a} + \\ \frac{K_A.A_2.R_S.K_S.F_m.N_2^{\ 2}}{J_L.J_m.L_a} \end{array} \right\} \end{array} \right]}$$

Using motor current feedback, one term has been added to the coefficient of s^3, two terms have been added to the coefficient of s^2, three terms have been added to the coefficient of s, and two terms have been added to the coefficient of "ω^4", in the transfer function denominators.

Taking the case of the motor driving a load via two reduction gearing stages, including a resilience element, but without any feedback applied, the denominator is fourth order in s (as page 9). Applying system values produced a numeric denominator bracket $\left[s^4 + s^3.73.03 + s^2.206762.7 + s.5218179.6 + 344339760.6\right]$.

Mathematica analysed the four roots as being $[-23.9152 - 450.884j],[-23.9152 + 450.884j],[-12.5998 - 39.1187j],[-12.5998 + 39.1187j]$.
Multiplying these as pairs gives two quadratics in s,
$\left[s^2 + s.47.83 + 203868.3\right] \times \left[s^2 + s.25.2 + 1689\right]$
So that $\xi_1 = 0.053,\ \omega_1 = 451.518,\ \xi_2 = 0.307,\ \omega_2 = 41.097$. The first of these quadratics is a motor transfer function, but the motor ω_n has increased from 366.11 rads/sec (The calculated value for the unloaded motor case), to 451.518, a result forced on the motor by the transfer function of the load. However, from the mathematics, the numeric value of the final term "ω^4" is unchanged, so that if the motor ω_n (ω_1) has increased by 23.33%, then ω_2 must have a similarly reduced value, since the term $\left[\omega_1.\omega_2\right]^2$ is constant. An investigation was carried out into the change in the motor ω_n (ω_1) in the motor denominator quadratic $\left[s^2 + s.47.83 + 203868.3\right]$ and the resulting changes to the "mechanical quadratic" ω value (ω_2) in the quadratic expression $\left[s^2 + s.25.2 + 1689\right]$ as either tacho or current feedback was applied, in increasing amounts from zero feedback level, but these results are not presented in this paper. The numerator "mechanical" quadratic $\left[s^2 + s.0.09822 + 2565.34\right]$ in the motor speed transfer function is not modified by feedback. The two mechanical quadratics (A numerator mechanical quadratic divided by a denominator mechanical quadratic) produce a typical resilience loop on a Nichols plot.

For a fourth-order-in-s denominator <u>only</u>, a second analytical approach can be used. Writing the two denominator quadratics as $\left[s^2 + s.2.\xi_1.\omega_1 + \omega_1^2\right] \times \left[s^2 + s.2.\xi_2.\omega_2 + \omega_2^2\right]$, and multiplying-out produces

$$\left[s^4 + s^3.\left\{2.\xi_1.\omega_1 + 2.\xi_2.\omega_2\right\} + s^2.\left\{\omega_1^2 + \omega_2^2 + 4.\xi_1.\omega_1.\xi_2.\omega_2\right\} + s.\left\{2.\xi_1.\omega_1.\omega_2^2 + 2.\xi_2.\omega_2.\omega_1^2\right\} + \left\{\omega_1^2.\omega_2^2\right\}\right]$$

It will be obvious that, if the value of either of the two damping coefficients needs to be increased considerably, such as wishing to increase the motor damping from around 0.1 to 1.0, so increasing the value of either ξ_1 or ξ_2, the term which must be significantly increased in the denominator is the coefficient of s^3, namely, $\left[2.\xi_1.\omega_1 + 2.\xi_2.\omega_2\right]$. This rule also applies to the transfer function denominators for more complex systems. The largest increase needs to be in the first coefficient of s that appears in the denominator. If the order-in-s of the denominator is, say, s^6, any feedback applied must cause a significant increase in the coefficient of s^5, and so on.

Using the coefficients for the various orders of s as in the numeric denominator expression given at the top of this page, four simultaneous equations can be written, which are
$\left[2.\xi_1.\omega_1 + 2.\xi_2.\omega_2\right] = 73.03$ ----(1), $\left[\omega_1^2 + \omega_2^2 + 4.\xi_1.\omega_1.\xi_2.\omega_2\right] = 206762.7$ ----(2),
$\left[2.\xi_1.\omega_1.\omega_2^2 + 2.\xi_2.\omega_2.\omega_1^2\right] = 5218179.6$ ----(3), $\left[\omega_1^2.\omega_2^2\right] = 344339760.6$ ----(4),

Mathematica will solve this equation set using the “Solve” command. The output is 24 sets of solutions
(Actually comprising 12 sets and 12 mirror-image sets, since ξ_1 can be interchanged with ξ_2 and ω_1 with ω_2 in the mathematics). Because the four terms $\xi_1, \omega_1, \xi_2, \omega_2$ are positive real numbers, the correct two solutions are obvious in the Mathematica printout, since all solutions with imaginary parts can be discounted. However, for 6th, and 8th, and order equations, only the “Roots” command can be used. These higher-order transfer functions generate sets of six, eight, or ten (etc) simultaneous equations, which Mathematica will not solve, there being too many possible permutations within the solution routine.

Also note that, when applying the “Solve” or “Roots” command to a transfer function numerator or denominator expression, the expression should be entered as an equation in x or in y, not as an equation in s. On seeing an equation in s as the input, Mathematica then looks for an InverseLaplaceTransform command, not for a “Roots” or “Solve” command.

(13)

Solutions using Mathematica “Roots” and “Solve” commands for the 4th order in s denominator expression produced identical quadratic solutions, as should obviously be the case.

The transfer function denominator expressions on pages 9,10,and 11, show that the use of motor tacho feedback adds one term to the coefficient of s^3, one term to the coefficient of s^2, and one term to the coefficient of s, but does not modify the last term (“ω^4”). The use of motor current feedback adds one term to the coefficient of s^3, two terms to the coefficient of s^2, three terms to the coefficient of s, and also adds two terms to the last term (“ω^4”). This indicates that, even when using tacho feedback, the ω_1 and ω_2 values are both changing as feedback is applied, and that the two feedback methods are modifying the parameters
ξ_1, ω_1, ξ_2, ω_2 differently, confirmed when numeric versions of the transfer function denominators are analysed by Mathematica.

The InverseLaplaceTransform functions generated by Mathematica will identify the frequency components present at the motor and load inertias (Two frequencies in the case under discussion, but, with three inertias present, four frequency components are extracted by Mathematica, three inertias plus the motor). The InverseLaplace Transform time-domain expression for $\left[Motor\ speed\right]_{(t)}$ is presented by Mathematica in the form

$$A.e^{-\xi_1.\omega_1.t}.e^{-\omega_1.\sqrt{\xi_1^2-1}.t} + B.e^{-\xi_1.\omega_1.t}.e^{\omega_1.\sqrt{\xi_1^2-1}.t} + C.e^{-\xi_2.\omega_2.t}.e^{-\omega_2.\sqrt{\xi_2^2-1}.t} + D.e^{-\xi_2.\omega_2.t}.e^{\omega_2.\sqrt{\xi_2^2-1}.t}$$
.

This is correct, but, since $\xi \le 1$ analysts use the form $\omega.\sqrt{1-\xi^2}.t$, and the above expression can be easily converted to this form, as below

$$A.e^{-\xi_1.\omega_1.t}.e^{-\omega_1.j\sqrt{1-\xi_1^2}.t} + B.e^{-\xi_1.\omega_1.t}.e^{\omega_1.j\sqrt{1-\xi_1^2}.t} + C.e^{-\xi_2.\omega_2.t}.e^{-\omega_2.j\sqrt{1-\xi_2^2}.t} + D.e^{-\xi_2.\omega_2.t}.e^{\omega_2.j\sqrt{1-\xi_2^2}.t}$$

But $e^{\omega_1.j\sqrt{1-\xi_1^2}.t} = \cos\left(\omega_1.\sqrt{1-\xi_1^2}.t\right) + j.\sin\left(\omega_1.\sqrt{1-\xi_1^2}.t\right)$ and

$e^{-\omega_1.j\sqrt{1-\xi_1^2}.t} = \cos\left(\omega_1.\sqrt{1-\xi_1^2}.t\right) - j.\sin\left(\omega_1.\sqrt{1-\xi_1^2}.t\right)$.

Similarly for the two $\left(\omega_2.\sqrt{1-\xi_2^2}.t\right)$ expressions.

The overall denominator expression therefore converts to the form

$$e^{-(\xi_1.\omega_1.t)}.\left[\left(A+B\right).\cos\left(\omega_1.\sqrt{1-\xi_1^2}.t\right)+\left(B-A\right).j.\sin\left(\omega_1.\sqrt{1-\xi_1^2}.t\right)\right]+e^{-(\xi_2.\omega_2.t)}.\left[\left(C+D\right).\cos\left(\omega_2.\sqrt{1-\xi_2^2}.t\right)+\left(D-C\right).j.\sin\left(\omega_2.\sqrt{1-\xi_2^2}.t\right)\right]$$

This contains two exponential damping coefficients, each associated with a different transient oscillating frequency term and multiplying / phase-shifting factor (The two frequencies have been identified, and their associated damping terms, also the phase-shifting coefficients A,B,C,D, which are complex). On plotting the numeric form of the above expression using Mathematica, identical plots were obtained as were obtained when using the original unmodified InverseLaplaceTransform expression, or when plotting motor speed / time responses using a full Matlab/Simulink system model.

It is claimed that it has been proved that the motor and load speed transfer function numerator and denominator expressions take the form of quadratics-in-s. For more complex systems, with two or three inertia elements, symbolic transfer functions were generated, and numeric versions of these transfer functions were calculated. Mathematica generated the sets of quadratics for both the numerator and denominator brackets for each of these more complicated cases. Time-response plots from Mathematica matched plots from full Matlab/Simulink system models of the two-inertia and three-inertia systems.

The analysis of the more complex systems was carried out partly in order to confirm the statements regarding the correct form of the transfer function expressions, also to derive a full set of transfer functions for a range of systems, from an unloaded motor, to a system with three inertia elements. Systems can usually be modelled accurately using one load inertia and one resilience element, as Figure 7, rather than using more complex models with two or three inertias, but, for some types of system, a two-inertia model must be used. Such systems include the stabilisation systems for turret-mounted tank guns, where the load inertia (Gun barrel) is outside the rate gyroscope feedback loop, the gyros being mounted on the gun cradle and not on the barrel itself. The gun then appears as a second inertia situated outside the rate or position transducer feedback point, coupled back to the position or rate pickoff by a second resilience element.

7. The Variation In Motor Frequency As The Load Transfer function Is Changed.

The single-inertia/single-resilience model (Figure 7) was used to investigate the change in motor frequency as the load parameters were varied. Transfer functions for this particular case were given on pages 9, 10, and 11 of the paper, for both the motor and load speeds, without any feedback, and with motor tacho and motor current

feedback applied. By varying the values of gear ratios N_1 and N_2, the resilience element can be located deliberately at any point within the reduction gearing. Throughout this particular investigation, the overall gear reduction was kept at 800:1. With $N_1 = 1$, and $N_2 = 800$, the resilience is positioned at the motor shaft.

With $N_1 = 800$ and $N_2 = 1$, the resilience is positioned at the load inertia. With the resilience placed at any defined position, the value of the resilience element stiffness figure can then be varied.

Numeric versions for the motor speed transfer function were calculated, with the resilience at the motor, at the load, and at five other positions within the geartrain, one having the values $N_1 = 46.24$ and $N_2 = 17.3$, the gear
ratios used in an actual system. At each position, the resilience stiffness figure K_S was varied over a 3-decade range, e.g., from 10 to 10^4 with the resilience at the motor and from 10^7 to 10^{10} with the resilience at the load. It was found that a three-decade range of K_S value generated a suitable range in motor ω_n variation to produce a good graph for all of the N_1 / N_2 ratio combinations used. For every setting of gear ratio and resilience stiffness figure, the numeric motor speed transfer function was calculated. Mathematica then extracted the two quadratics-in-s in the motor speed transfer function denominator, using the “Roots” command.

One of these represented the modified motor quadratic, the second denominator quadratic producing the low-frequency (mechanical) component in the motor speed plot that can be seen in Figure 8.

Refer to Figure 15. With the resilience situated at the motor, as K_S tends to zero, the motor performs like a totally unloaded motor, with ω_n = 366.11. As K_S is increased from the value of 10 Nm/radian to 2500 Nm/radian, the motor frequency increases, relatively slowly at first, then more quickly. With K_S = 4000, the motor ω_n figure has increased to 1500 rads./sec. from 366.11 rads./sec. Plot 1 plots the results calculated by Mathematica with the resilience placed at the motor. Plots 2, 3, etc are similar plots with the resilience located within the geartrain, but further from the motor. In Figure 15, Plot 1, through to Plot 7, are identical curves. It is seen that, as the value of K_S is increased towards the top-end of the range being considered, the motor frequency eventually increases rapidly, and without limit (confirmed by running the Matlab/Simulink model, Figure 7). If K_S is increased over a very wide range indeed, even a powerful 3 Ghz computer will run out of computing power, since the numbers being processed rapidly become exceedingly large, and the model will not run.

A horizontal line has been drawn on Figure15, through the +20% increase in motor natural frequency figure ω_n, the reason being that the slope of each of the plots is large enough for K_S values to be read accurately, plotted at A3 size. It is seen that, to have the same effect on motor ω_n, Plot 2 is effectively Plot 1, but moved along the x-axis by a factor of N_1^2 (When N_1 = 5, Plot 1 moves to the right by a factor of 25, becoming Plot 2. When N_1 = 20, Plot 1 moves to the right by a factor of 400, becoming Plot 3, etc.). Plot 7 is defined by multiplying Plot 1 values by 800^2, or $6.4x10^5$. Point A (K_S value = 107) moves to [107x $6.4x10^5$] or to the value K_S = [$6.848x10^7$], Point G. Conversely, considering Plot 7, with the resilience at the load, moving from Plot 7 to Plot 6 is achieved by dividing the K_S values for Plot 7 by N_2^2, and so on.

8. Summary And Conclusions.

The writer has carried out the analysis work described, regarding the correct form of transfer functions for motor speed and for load speed, in systems containing one, two, and three load inertias. A list of the rules derived has been presented, and a description of the methods used to confirm these, using Mathematica software and Matlab/Simulink system models. It has been shown that the form of equation proposed by Dr. A. L. Whiteley is not correct, which makes papers and textbook chapters based on this form also incorrect. It is stated that, in order to generate the various oscillation frequencies and damping terms, within a system containing one or more load-inertia elements, the correct form for the numerator and denominator expressions in these transfer functions comprises sets of simple quadratics in s, multiplied together. The general concept of “Standard form” expressions has been shown to be untenable by the writer’s analysis.

The work has shown that the order of the transfer function numerator is always two orders less than the denominator, at the motor, and that one numerator quadratic is dropped as the point of analysis moves from inertia to inertia, away from the motor. It has also been shown that the numerator quadratics are different, in both frequency and damping content, at each of the inertia elements (The numerator quadratic expressions change completely, when moving from inertia to inertia. It is not simply a case of one quadratic being eliminated, with the rest being unchanged). The denominator expression, which can comprise two, three, or four quadratic expressions, depending on the drive arrangement loading the motor, remains constant throughout the analysis.

Matlab/Simulink models, both fully-variable, as Figures 2 and 7, and simpler transfer function models, as
Figure 5, have also been generated for systems with two and three inertias. The symbolic mathematical transfer functions for all of these systems have been calculated. Numeric versions have been derived and have been

analysed using Mathematica. Results from both types of Matlab/Simulink model have been compared with results from Mathematica. These results included the use of tacho and current feedback, and cases without any feedback applied.

This paper has described the analysis work and the most significant results. The work took some three years to complete, and was started in September of 2003, after completing some additional analysis work on stabilisation systems and preparing the AECV2003 Symposium paper given at Angers in June 2003. A later AECV2005 paper [5], given at Bath in June 2005, presented some results from the analysis, as the work stood at the time.

In the U.K., accurate stabilisation and position systems have always applied some form of negative-acceleration minor loop feedback, but it is claimed that the subject had previously not been investigated adequately. The work carried out by the writer, described in this paper, enables the subject to be better understood. The differences between tacho and current feedback have been investigated and some results presented. Current feedback produces calculable and improved load-inertia damping and clearly does not require the fitting of a permanent-magnet tacho to the motor. At the same time, the peak motor current (Hence the peak motor power and also peak motor controller power) reduces significantly, as do the resulting stress levels within the mechanics of the system.

The results from the analysis state clearly that the general concept of “Standard form” denominator expressions is not feasible, that the form of equations presented, (e.g. in the Whiteley thesis, and in control system papers and textbooks), is not correct, also that odd-order equations are not possible, in physically-realisable systems. The correct form of transfer function expressions for a wide range of systems have been calculated and analysed, confirming the writer’s results.

10. Ongoing Analysis Work.

Work is still ongoing regarding the application of the mathematical analysis work, and computer modelling, to rate-gyroscope systems, also to precise position systems. Figure 16 has been included to illustrate the one-inertia / one-resilience system of Figure 7 operating within a rate-gyro outer loop. This model is fully-compensated for the fact that the system does not have infinite loop gain, for motor friction, and for load friction. The acceleration feedback used in the model is motor tacho, but the use of motor current feedback should produce the system benefits discussed earlier, and will be investigated. As it stands at present, this model produces an extremely accurate stabilisation performance at the load inertia.

List of definitions used in the system analysis.

A_2 The voltage gain of the amplifier used to scale either the tacho feedback or the current feedback signal, when these feedback voltages are used to modify the motor-speed and load-speed transfer functions, by applying a negative acceleration feedback term.

F_L Load viscous friction. Nm/radian/second, but given in Nm/degree/second in some of the author’s Matlab/Simulink models.

F_m Motor viscous friction. Always defined in Nm/radian/second.

I_a Motor current. Amps.

J_L Load inertia, kg.m^2. (J_{L1}, J_{L2}, etc.)

J_m Motor rotor inertia, kg.m^2. Includes the inertia of the first stage of reduction gearing, referred to the shaft of the servo motor.

K_A The voltage gain of the motor controller unit driving the servo motor.

K_T The motor torque constant, Nm/amp.

K_V The motor speed constant, volts / radian / second.

R_a Motor resistance, Ohms.

L_a Motor inductance, Henries.

K_S The stiffness of any system resilience element, defined in Nm / radian deflection.

N In systems not containing a resilience term, with a single gear reduction ratio, N is the total reduction ratio of the servo gearbox.

N_1, N_2 Gear reduction ratios. In systems containing a single resilience element, the reduction gearing is

divided into sections N_1 and N_2.
V_{IN} The input voltage to the motor controller unit.
V_{OUT} The output voltage from the motor controller unit.
θ_m The position of the motor shaft, radians.
θ_L The position of the load inertia. This can be identified either in degrees or in radians.

References.

1. AECV2003. Conference at Angers. France. June 2003. Paper by Dr. F. Garner.
2. Thesis for D.Sc. A. L. Whiteley. 1945.
3. Modern Control System Theory And Design. S.M. Shinners. J. Wiley & Sons. Page 295.
4. The Synthesis Of “Optimum” Transient Response. Criteria And Standard Forms. Graham & Lathrop. AIEE Summer General Meeting. Atlantic City. June 1953.
5. AECV2005. Conference at Bath. U.K. Paper by Dr. F. Garner.

Dr. F. Garner.
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