Multi-objective Controller Design:

Evolutionary algorithms and Bilinear Matrix Inequalities for a passive suspension

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Linear Matrix Inequalities

A linear matrix inequality (LMI) is an expression,

$$F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n > 0$$

- $F_i = F_i^T$ are real symmetric matrices,
- The inequality means that $F(x)$ is positive definite matrix

LMI example problem:

$$F(x_1, x_2) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} > 0$$

this is equivalent to

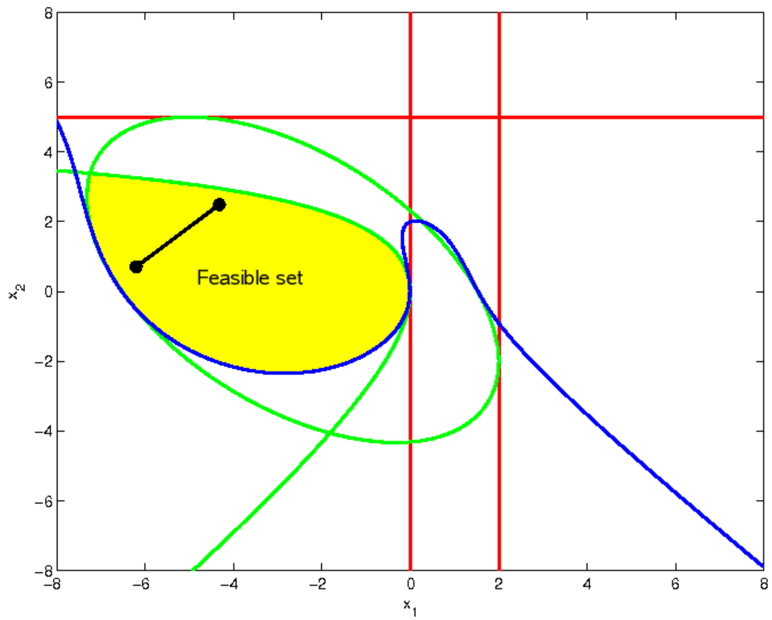
$$F(x_1, x_2) = \begin{bmatrix} 2 - x_1 & -(x_1 + x_2) & 0 \\ -(x_1 + x_2) & 5 - x_2 & x_2 \\ 0 & x_2 & -x_1 \end{bmatrix} > 0$$

Geometry of LMIs (Convexity)

$$F(x_1,x_2) = \begin{bmatrix} 2-x_1 & -(x_1+x_2) & 0 \\ -(x_1+x_2) & 5-x_2 & x_2 \\ 0 & x_2 & -x_1 \end{bmatrix} > 0$$

- Feasible iff all principal minors are nonnegative polynomial inequalities $f_i(x) > 0$

$$\begin{aligned} f_1(x) &= 2 - x_1 > 0, \\ f_2(x) &= 5 - x_2 > 0, \\ f_3(x) &= -x_1 > 0, \\ f_4(x) &= (2-x_1)(5-x_2)- \\ &\quad -(x_1+x_2)^2 > 0, \\ f_5(x) &= x_1(2-x_1) > 0, \\ f_6(x) &= -x_1(5-x_2) - x_2^2 > 0, \\ f_7(x) &= -x_1((2-x_1)(5-x_2)- \\ &\quad -(x_1+x_2)^2) - x_2^2(2-x_1) > 0 \end{aligned}$$



Intersection of LMIs

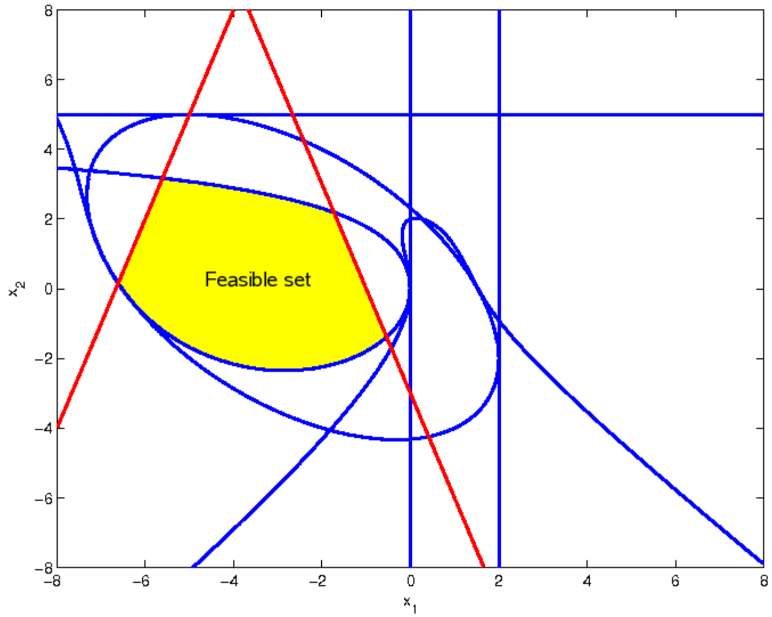
- If $F(x)$ and $G(x)$ are LMIs, the intersection has the form

$$H(x) = \begin{bmatrix} F(x) & 0 \\ 0 & G(x) \end{bmatrix} > 0$$

- Example of intersections

$$F(x) = \begin{bmatrix} 2 - x_1 & -(x_1 + x_2) & 0 \\ -(x_1 + x_2) & 5 - x_2 & x_2 \\ 0 & x_2 & -x_1 \end{bmatrix} > 0$$

$$G(x) = \begin{bmatrix} 3x_1 + x_2 + 3 & 0 \\ 0 & -3x_1 + x_2 - 20 \end{bmatrix} > 0$$



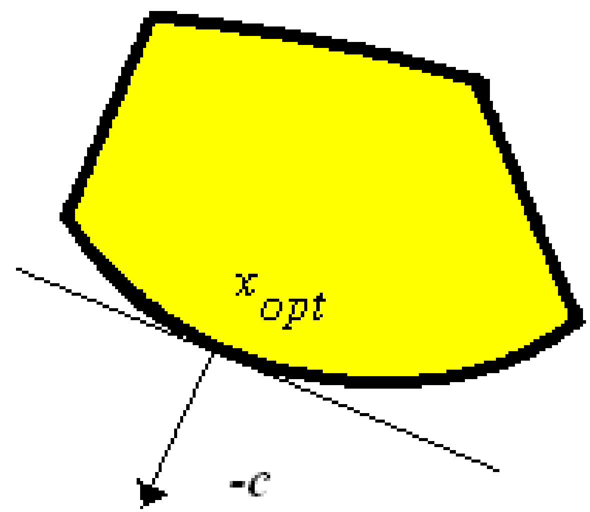
Convex optimisation over Linear Matrix Inequalities (LMIs) / Semidefinite Programming (*SDP*)

An *SDP* is an optimisation problem with ***linear objective*** and ***semidefinite constraints*** or LMI constraints.

$$\min_{x \in \Psi} \quad c^T x$$

$$subject \quad to \qquad F(x) = F_O + \sum_{i=1}^{m} x_i F_i > 0$$

In 1994, the ***LMI solvers*** became widely available. i.e. Projective Algorithm of Nesterov and Nemirovski (***interior-point methods***), available in the LMI Control toolbox.



In control theory, the idea is to reduce a multiobjective control problem into a ***convex*** optimisation or *SDP* problem.

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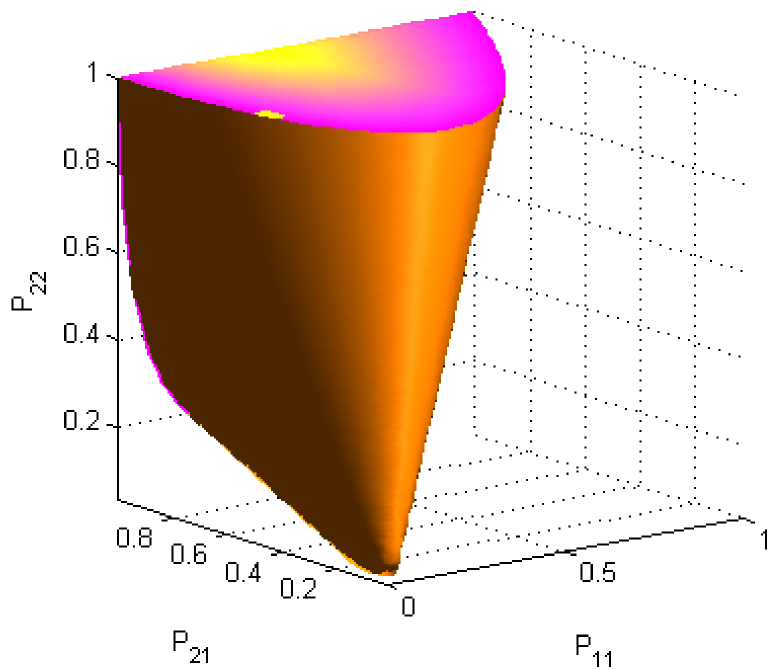
Lyapunov's Inequality (1892)

The system $\dot{x} = Ax$
is asymptotically stable if there exist a matrix *P>0* (positive definite) that satisfied the Lyapunov inequality

$$A^T P + PA < 0,$$



$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}^T \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} + \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} < 0$$

Henrion, 2003

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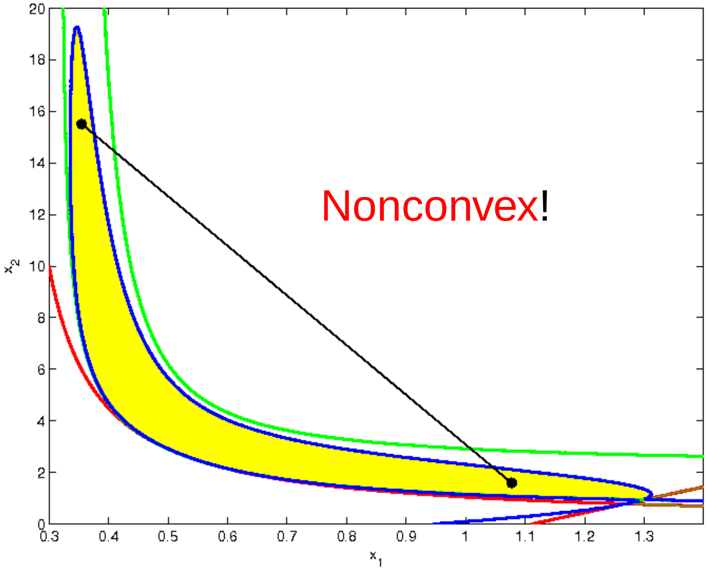
Bilinear Matrix Inequalities

A bilinear matrix inequality (BMI) is an expression,

$$F(x) = F_0 + \sum_i x_i F_i + \sum_i \sum_j x_i x_j F_{ij} < 0$$

BMI example (Henrion,2005):



$$F(x) = \begin{bmatrix} -10 & -0.5 & -2 \\ -0.5 & 4.5 & 0 \\ -2 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} 9 & 0.5 & 0 \\ 0.5 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix} +$$

$$x_2 \begin{bmatrix} -1.8 & -0.1 & -0.4 \\ -0.1 & 1.2 & -1 \\ -0.4 & -1 & 0 \end{bmatrix} + x_1 x_2 \begin{bmatrix} 0 & 0 & 2 \\ 0 & -5.5 & 3 \\ 2 & 3 & 0 \end{bmatrix} < 0$$

Mixed H_2/H_∞ Control

State-feedback close-loop system with two performance channels

$$\begin{aligned} \dot{x} &= (A + B_w K)x + B_w w \\ z_\infty &= (C_\infty + D_{\infty u} K)x + D_{\infty w} w \\ z_2 &= (C_2 + D_{2u} K)x \end{aligned}$$

and mixed performance specification

$$||T_{w\to z_\infty}||_\infty < \gamma_\infty \quad \text{and} \quad ||T_{w\to z_2}||_2 < \gamma_2,$$

BMIs for the H_∞ norm constraint

$$\begin{bmatrix} AX_\infty + B_u K X_\infty + (*) + B_w B_w^T & * \\ C_\infty X_\infty + D_{\infty_u} K X_\infty + D_{\infty w} B_w^T & D_{\infty w} D_{\infty w}^T - \gamma_\infty^2 I \end{bmatrix} < 0, \quad X_\infty > 0$$

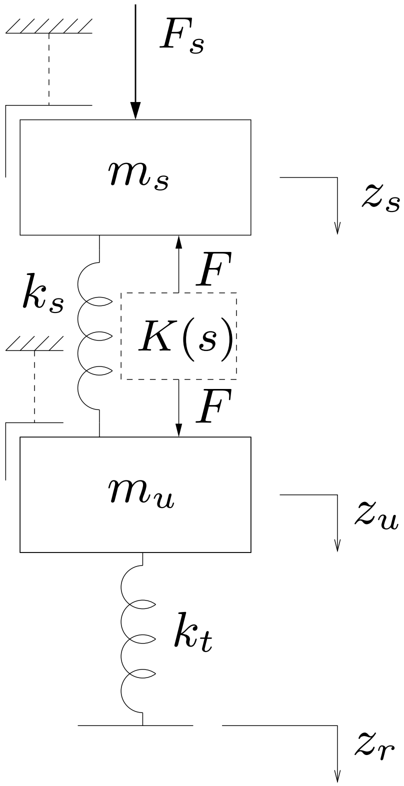
BMIs for the H_2 norm constraint

$$\begin{bmatrix} AX_2 + B_u K X_2 + (*) & B_w \\ * & -I \end{bmatrix} < 0, \qquad \begin{bmatrix} W & C_2 X_2 + D_{2u} K X_2 \\ * & X_2 \end{bmatrix} > 0, \ \text{tr}(W) < \gamma_2$$

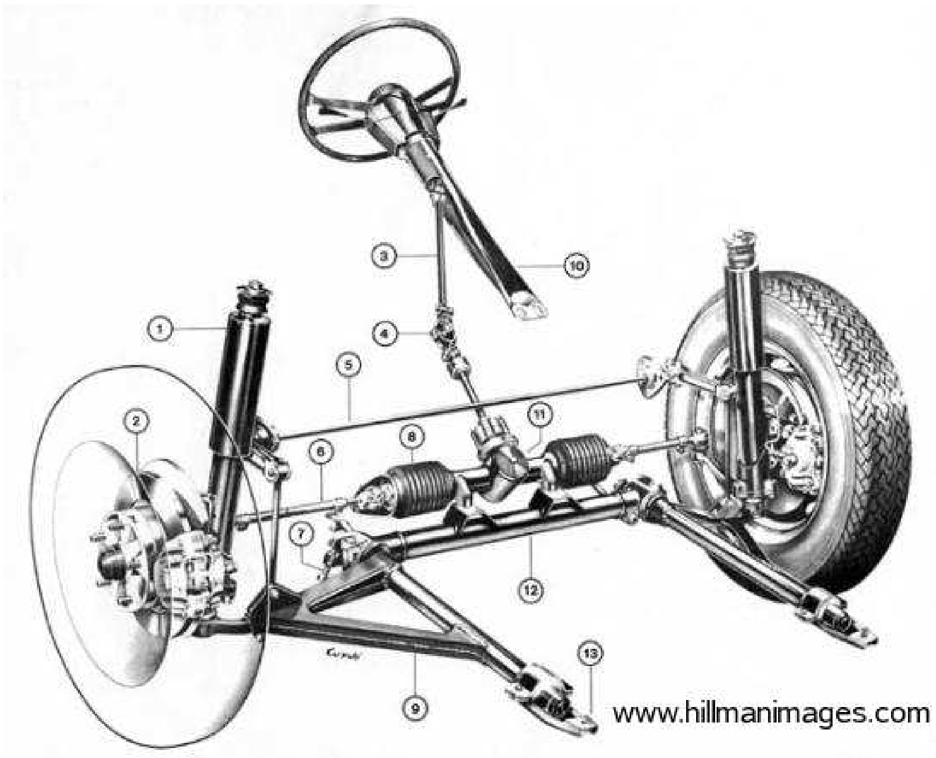
Two terms KX_∞ and KX_2 cannot be linearized simultaneously, Remedy $X = X_\infty = X_2$, then $Y = KX$ as change of variable to recover convexity, this leads to conservative results.

Mechanical network: vehicle suspension

quarter-car model



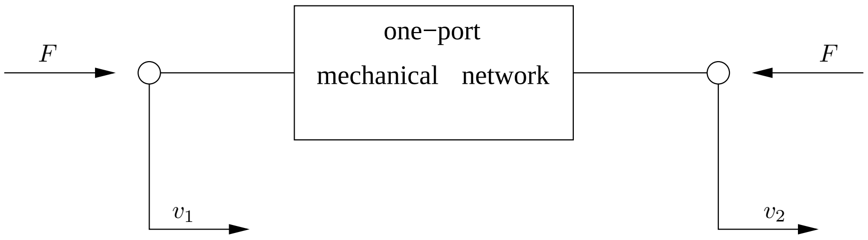
Front suspension



Mechanical networks

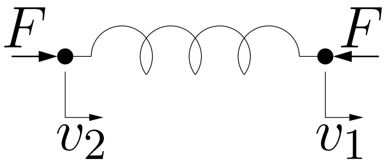
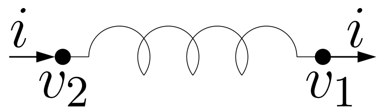
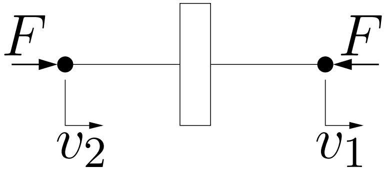
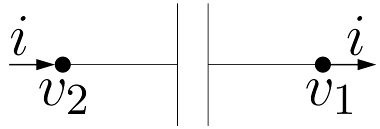
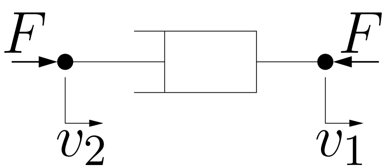
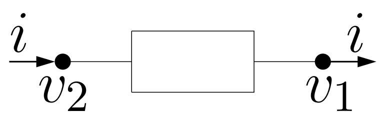
- A mechanical network is a rigid interconnection of mechanical elements (springs, masses, dampers and levers).
- A mechanical one-port network with force-velocity pair (F, v) is defined to be passive if,

$$\int_0^T F(t)v(t)dt \geq 0.$$



- Impedance is defined as a positive-real function $Z(s) = \frac{\text{across variable}}{\text{through variable}} = \frac{\hat{v}}{\hat{F}}$
- Admittance is defined as a positive-real function $Y(s) = \frac{1}{Z(s)} = \frac{\hat{F}}{\hat{v}}$.

Force-current analogy

Mechanical	Electrical
 $Y(s) = \frac{k}{s}$ $\frac{dF}{dt} = k(v_2 - v_1)$ spring	 $Y(s) = \frac{1}{Ls}$ $\frac{di}{dt} = \frac{1}{L}(v_2 - v_1)$ inductor
 $Y(s) = bs$ $F = b\frac{d(v_2 - v_1)}{dt}$ mass?	 $Y(s) = Cs$ $i = C\frac{d(v_2 - v_1)}{dt}$ capacitor
 $Y(s) = c$ $F = c(v_2 - v_1)$ damper	 $Y(s) = \frac{1}{R}$ $i = \frac{1}{R}(v_2 - v_1)$ resistor

The Inerter

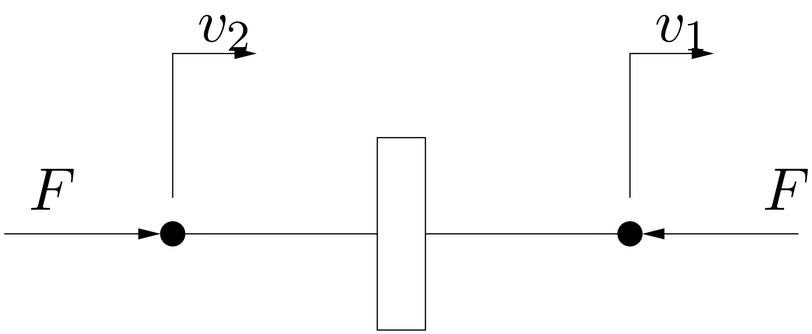
Concept: A mechanical two-terminal device such that the relative acceleration between the terminals is proportional to the force applied at the terminals.

- M. C. Smith, Synthesis of Mechanical Networks: The Inerter, IEEE Transactions on Automatic Control, $47(10)$, $1648 - 1662$, 2002

The Inerter

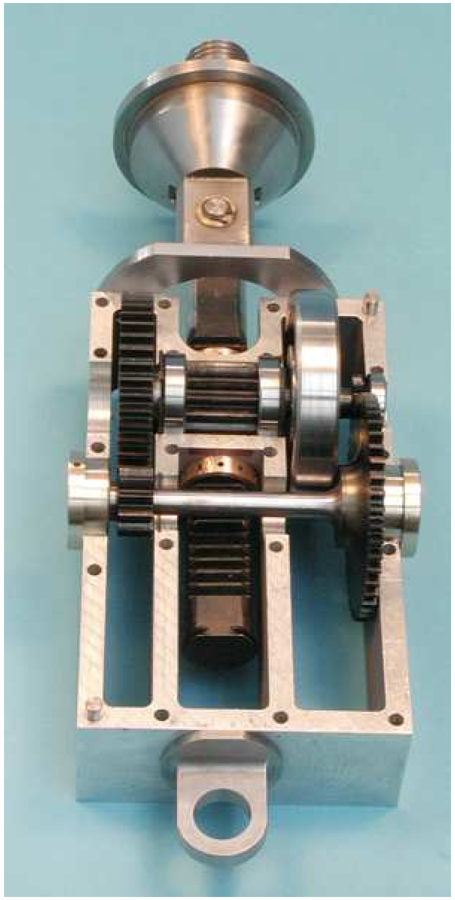
Concept: A mechanical two-terminal device such that the relative acceleration between the terminals is proportional to the force applied at the terminals.

Theory:



$$F = b(\dot{v}_2 - \dot{v}_1)$$

b : inertance

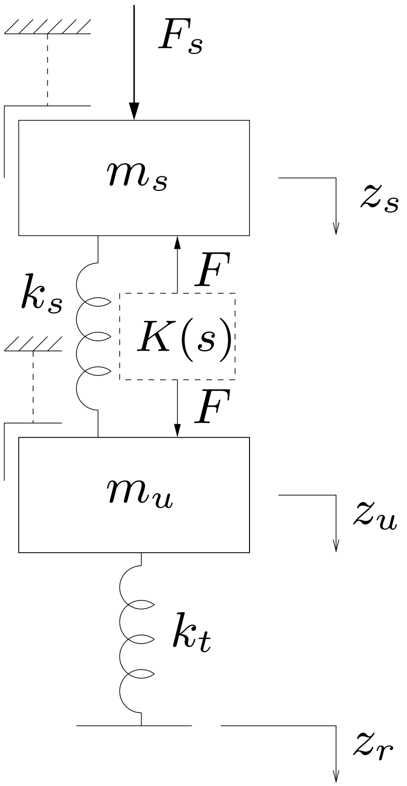


Practice: Can one build an inerter?

Rack-and-pinion inerter, mass ≈ 2 kg,

inertance $= 70 - 700$ kg, travel $= 80$ mm

Suspension for the quarter-car model



sprung and unsprung mass

$$\ddot{z}_s = \frac{F_s}{m_s} - \frac{F}{m_s} - \frac{k_s}{m_s}(z_s - z_u),$$

$$\ddot{z}_u = \frac{F}{m_u} + \frac{k_s}{m_u}(z_s - z_u) + \frac{k_t}{m_u}(z_r - z_u),$$

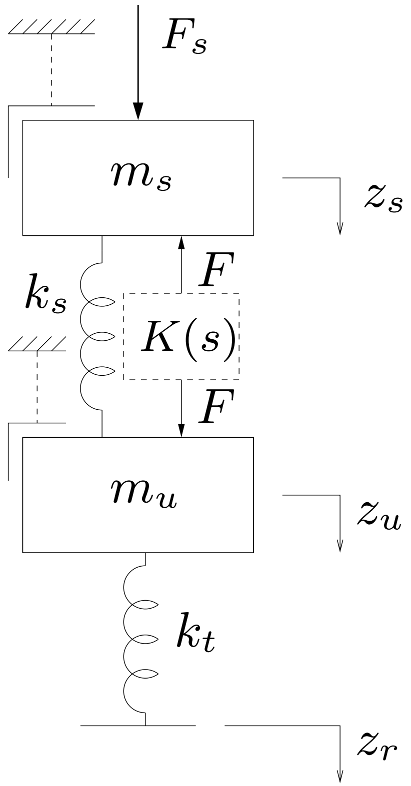
suspension

$$\hat{F} = K(s)(s\hat{z}_s - s\hat{z}_u),$$

Ride comfort: $J_1 := 2\pi(V\kappa)^{(1/2)}||sT_{\hat{z}_r \to \hat{z}_s}||_2$

RMS body vertical acceleration in response to road disturbances

Suspension for the quarter-car model



sprung and unsprung mass

$$\begin{aligned}\ddot{z}_s &= \frac{F_s}{m_s} - \frac{F}{m_s} - \frac{k_s}{m_s}(z_s - z_u),\\ \ddot{z}_u &= \frac{F}{m_u} + \frac{k_s}{m_u}(z_s - z_u) + \frac{k_t}{m_u}(z_r - z_u),\end{aligned}$$

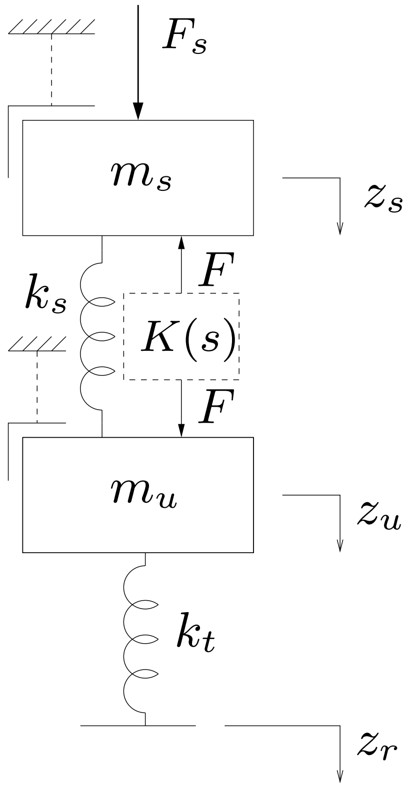
suspension

$$\hat{F} = K(s)(s\hat{z}_s - s\hat{z}_u),$$

Tyre grip: $$J_3 := 2\pi(V\kappa)^{1/2} \left\| \frac{1}{s} T_{\hat{z}_r \to k_t(\hat{z}_u - \hat{z}_r)} \right\|_2$$

RMS dynamic tyre load in response to road disturbances

Suspension for the quarter-car model



sprung and unsprung mass

$$\ddot{z}_s = \frac{F_s}{m_s} - \frac{F}{m_s} - \frac{k_s}{m_s}(z_s - z_u),$$

$$\ddot{z}_u = \frac{F}{m_u} + \frac{k_s}{m_u}(z_s - z_u) + \frac{k_t}{m_u}(z_r - z_u),$$

suspension

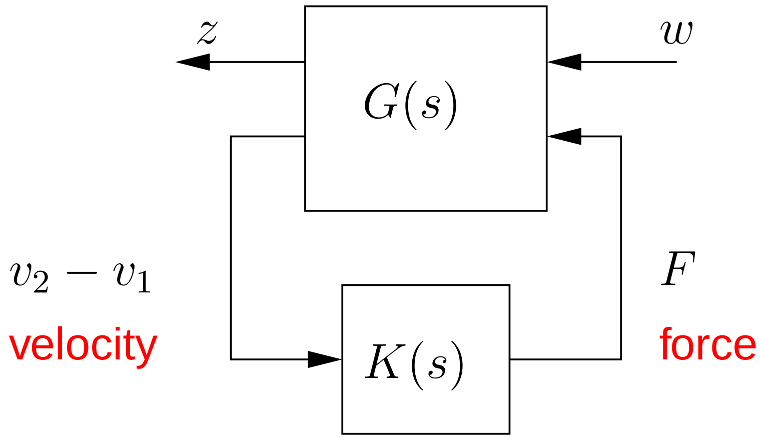
$$\hat{F} = K(s)(s\hat{z}_s - s\hat{z}_u),$$

Rejection of external loads: $J_5 := \left\| T_{\hat{F}_s \to \hat{z}_s} \right\|_\infty$

Control problem formulation

- Objective: Synthesize a positive real admittance $Y(s)$ to improve performance criterions.

- Formulation: Optimize a vector of $||T_{w \to z}||$ over positive real controllers $K(s)$.



- Solution: Characterize and solve the problem using Linear Matrix Inequalities

LMI Formulation

- Characterize $\mathscr{H}_\infty$ and $\mathscr{H}_2$ performances (Scherer et al., 1997),
  - $||T_{w\to z}||_2^2 = Tr(CXC^T)$, X solves a Lyapunov equation ...
  - Bounded real lemma of $||T_{w\to z}||_\infty = \sup_{w\in\mathscr{L}_2} \frac{||z||_2}{||w||_2}$ ...
- Positive Real Lemma (Boyd et al., 1994): Given that $K(s)$ positive real,

$$\exists\, X > 0, \begin{bmatrix} A^T X + XA & XB - C^T \\ B^T X - C & -D^T - D \end{bmatrix} \leq 0$$

multi-objective controller design

- Simultaneous J_1 and J_3 minimization,

$$\min_{K(s) \text{ positive real}} ||T_{\hat{z}_r \rightarrow \hat{\dot{z}}_s}||_2 \ and \ |||T_{\hat{z}_r \rightarrow \int(\hat{z}_u - \hat{z}_r)}||_2$$

- Positive real constraint $\rightarrow$ bilinear matrix inequality with respect to Lyapunov matrices X_{cl}, X_k and controller matrix $K(s)$
- The approach taken here is to minimise

$$\sqrt{(1-\lambda)\frac{J_1^2}{\hat{J_1}^2} + \lambda\frac{J_3^2}{\hat{J_3}^2}}, \quad for \ \ 0 < \lambda < 1$$

- Solved locally with iterative convex optimization methods

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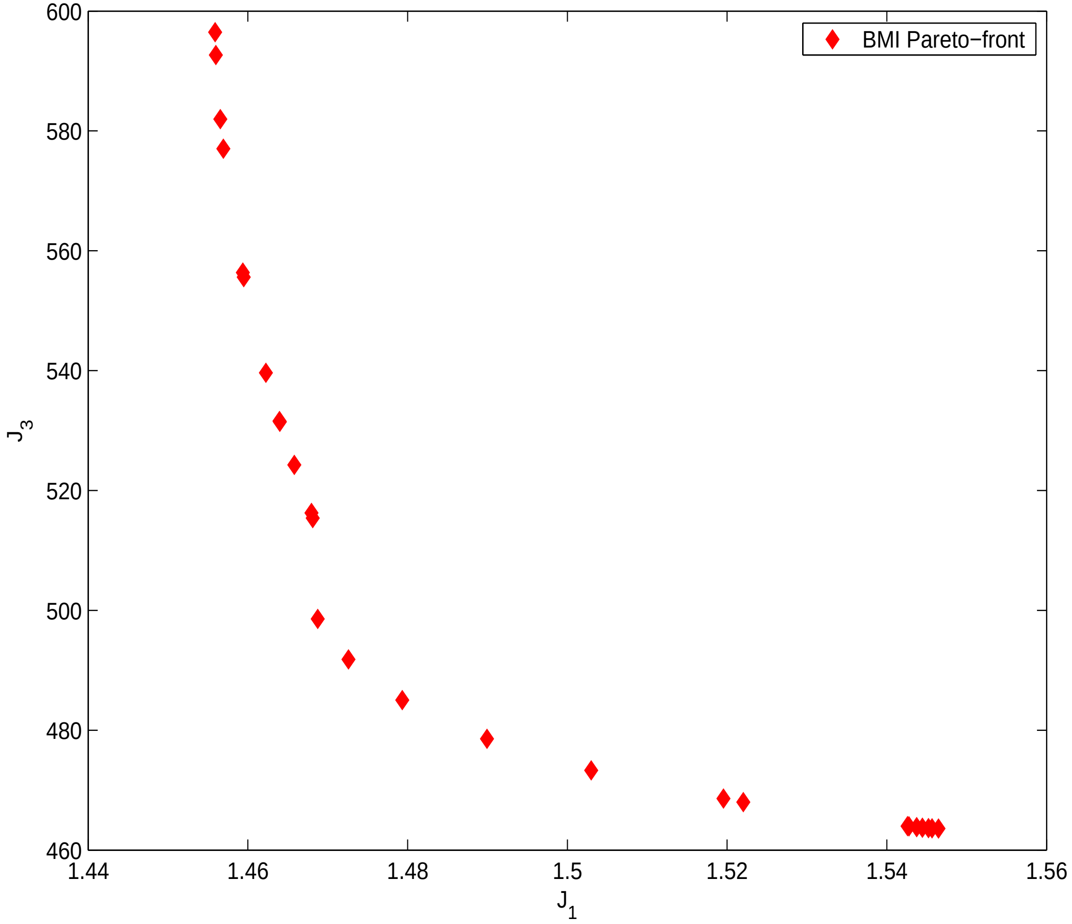
Multi-objetive optimisation with BMIs



Lolcal Search: Relies on the intuitive choice of a feasible starting point

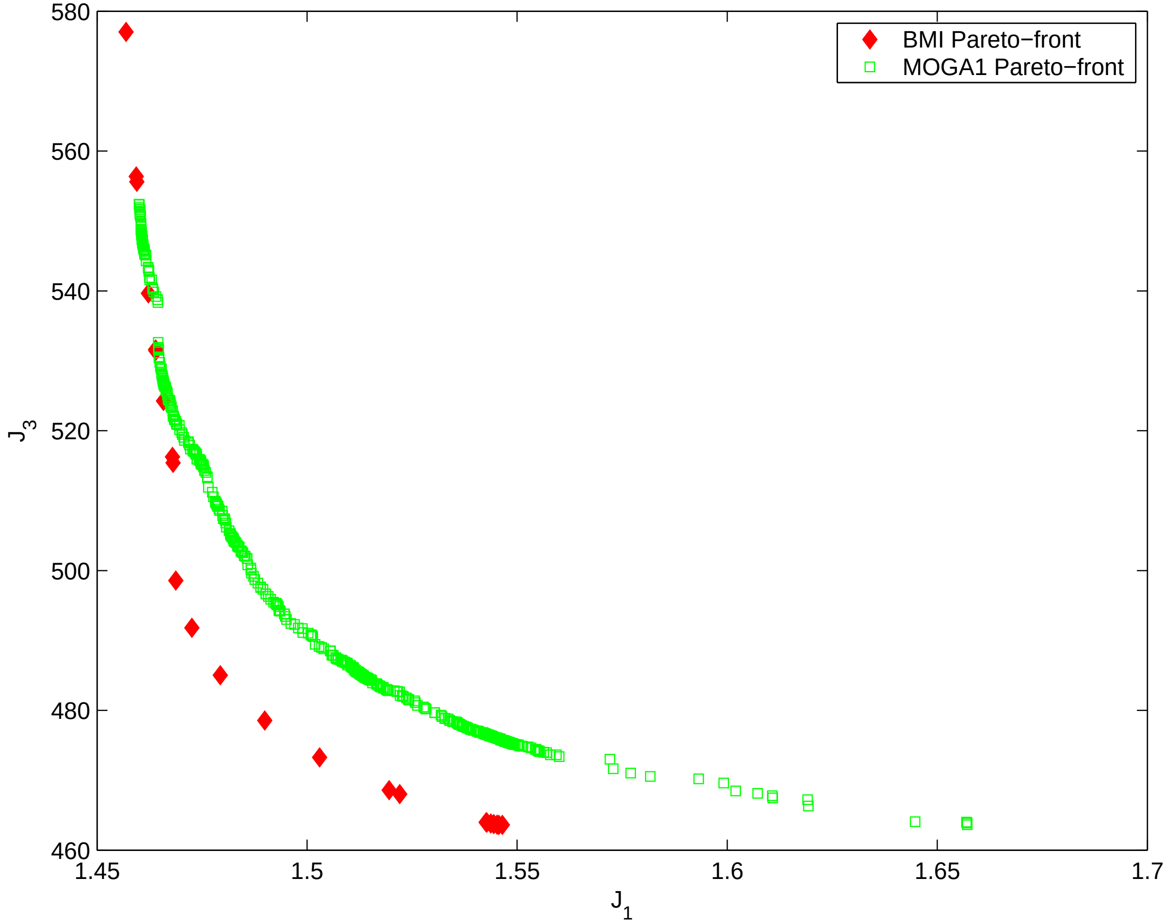
MOGA-based method

- Parameter encoding: The decision variables $\rightarrow$ controller

$$K(s) = bs + c\frac{s^2 + a_1 s + a_2}{s^2 + b_1 s + b_2}$$

- Optimise simultaneously:
  - Ride comfort $J_1 := 2\pi(V\kappa)^{(1/2)}||sT_{\hat{z}_r \rightarrow \hat{z}_s}||_2$
  - Tyre grip $J_3 := 2\pi(V\kappa)^{1/2}\left\|\frac{1}{s}T_{\hat{z}_r \rightarrow k_t(\hat{z}_u - \hat{z}_r)}\right\|_2$
  - subject to $K(s)$ been positive real (constraint)

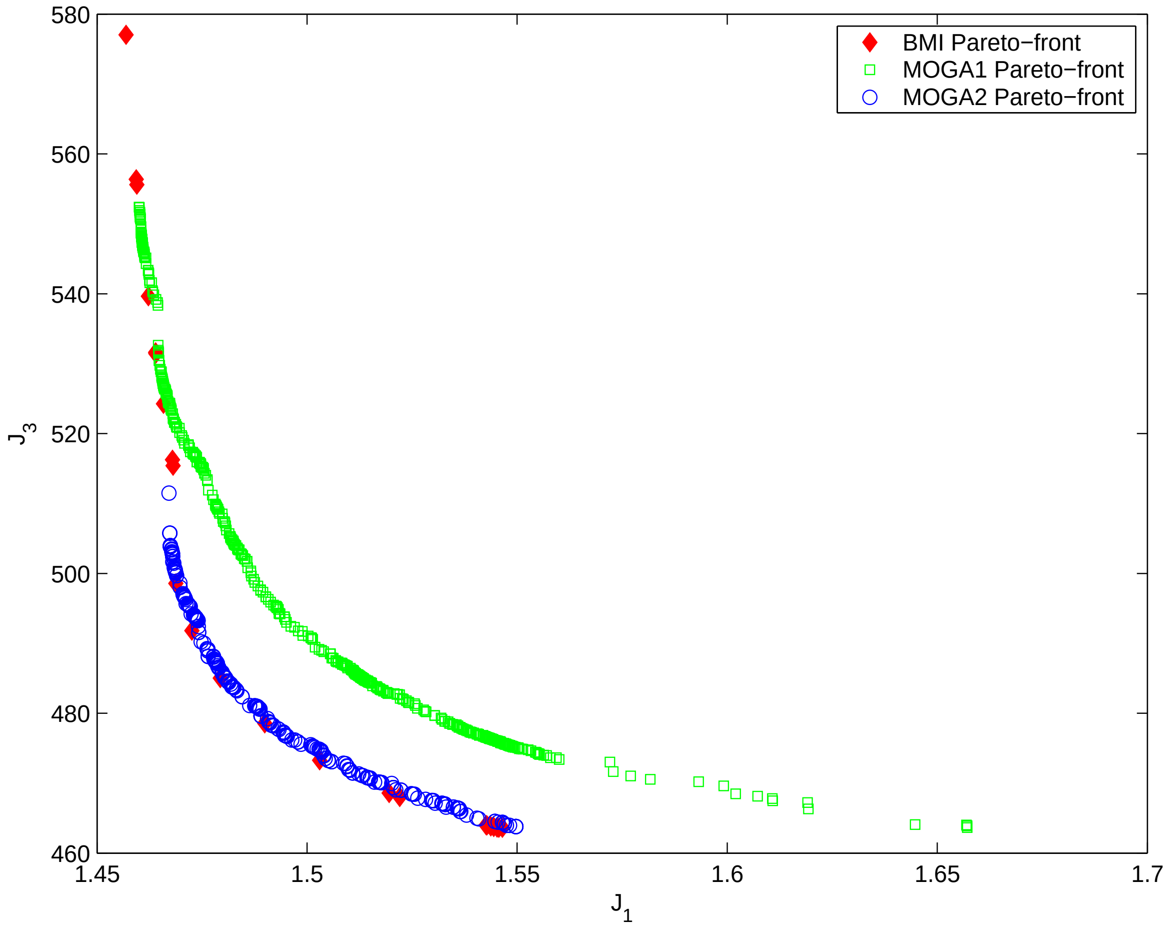
Multi-objetive optimisation with GAs



Converge to a Local Pareto optimum

Deceptive problem: often the entire search favors the non-global optimum

Multi-objective optimisation with GAs



Remedy: Increase the population size (from 200 to 600 individuals)

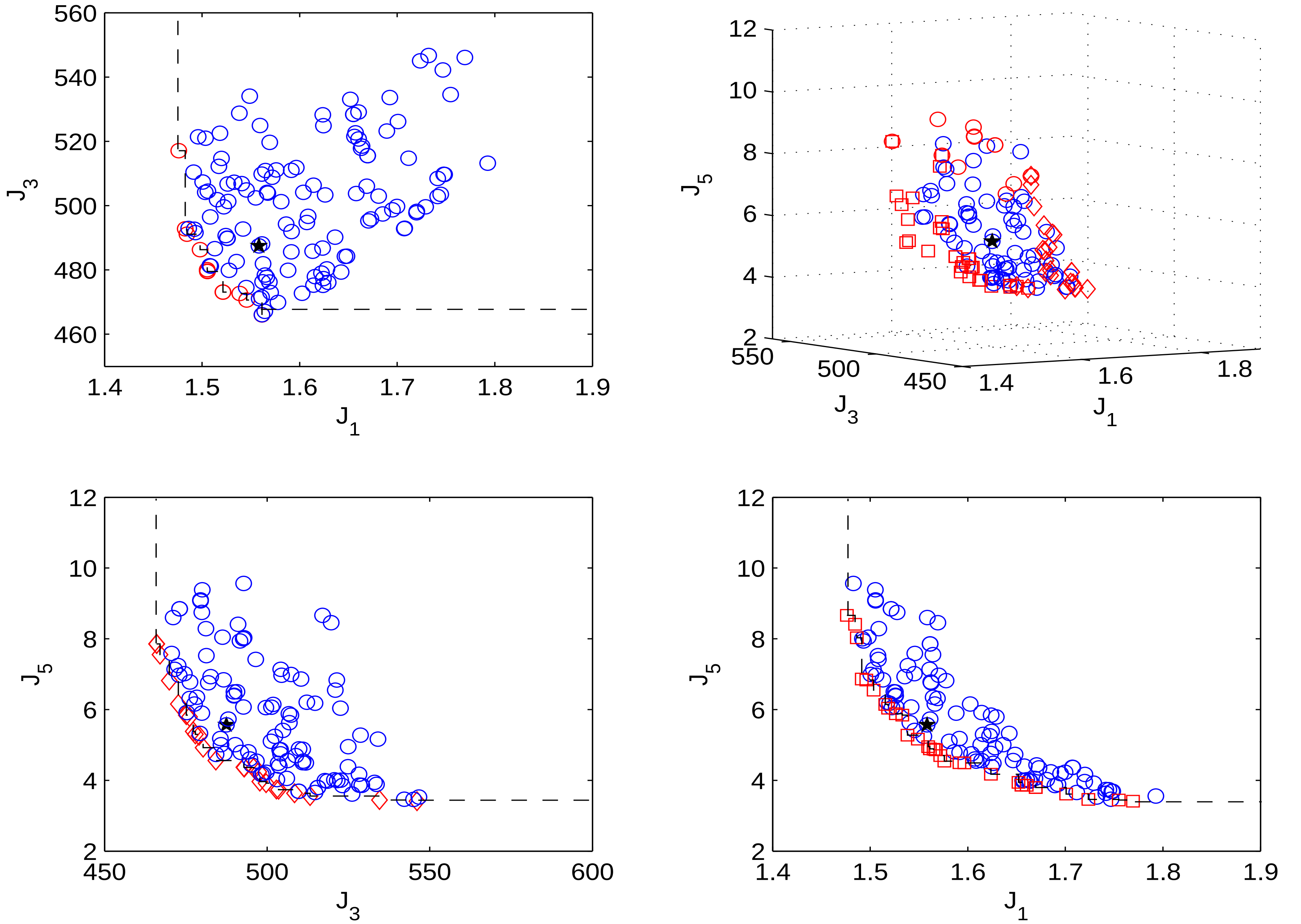
MOGA-based method:three objectives

- Parameter encoding:The decision variables $\rightarrow$ controller

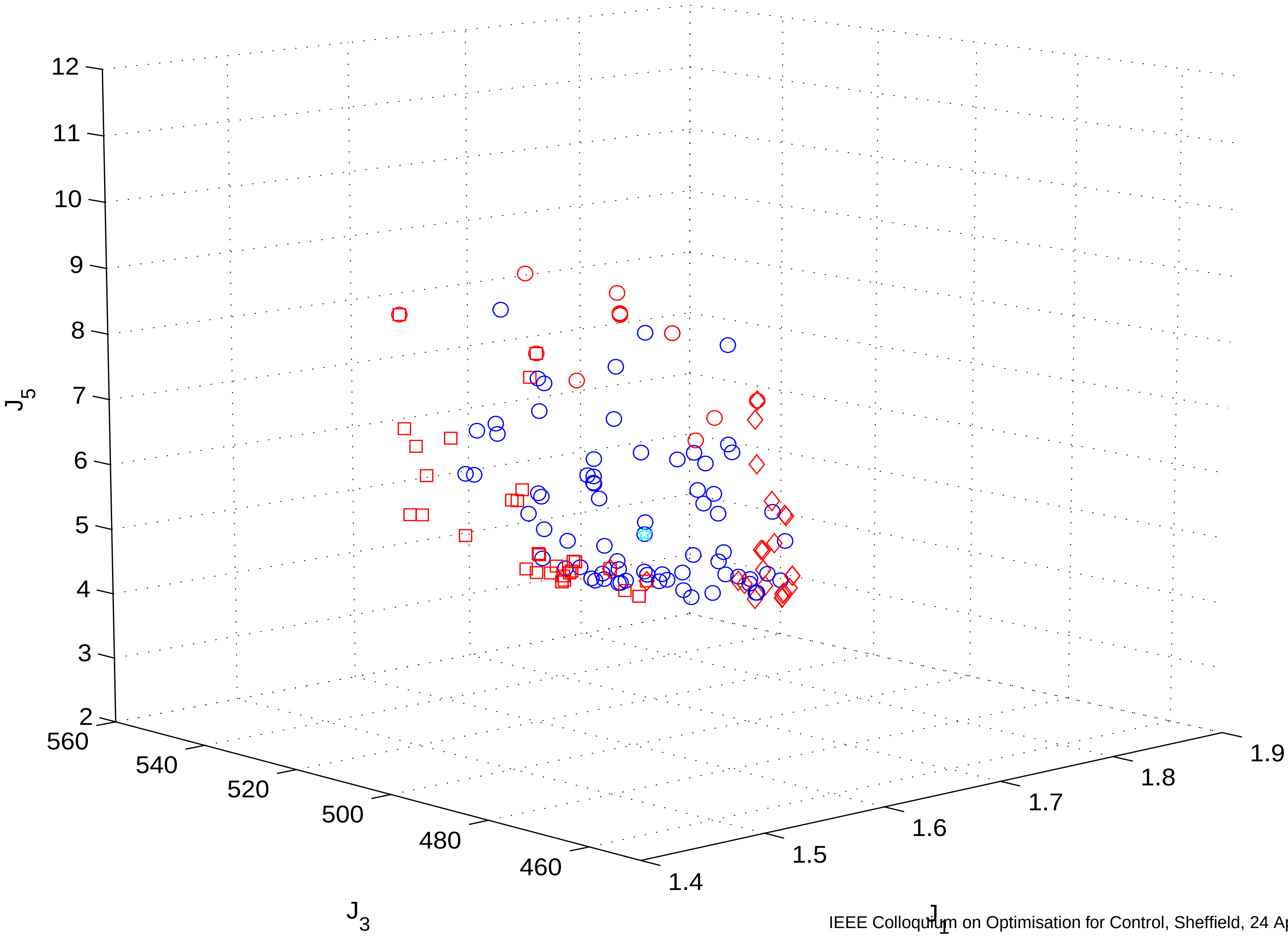
$$K(s) = bs + c\frac{s^2 + a_1 s + a_2}{s^2 + b_1 s + b_2}$$

- Optimise simultaneously:
  - Ride comfort: $J_1 := 2\pi(V\kappa)^{(1/2)}||sT_{\hat{z}_r \rightarrow \hat{z}_s}||_2$
  - Tyre grip: $J_3 := 2\pi(V\kappa)^{1/2}\left\|\frac{1}{s}T_{\hat{z}_r \rightarrow k_t(\hat{z}_u - \hat{z}_r)}\right\|_2$
  - Rejection to external roads: $J_5 := \left\|T_{\hat{F}_s \rightarrow \hat{z}_s}\right\|_\infty$
  - subject to $K(s)$ been positive real (constraint)

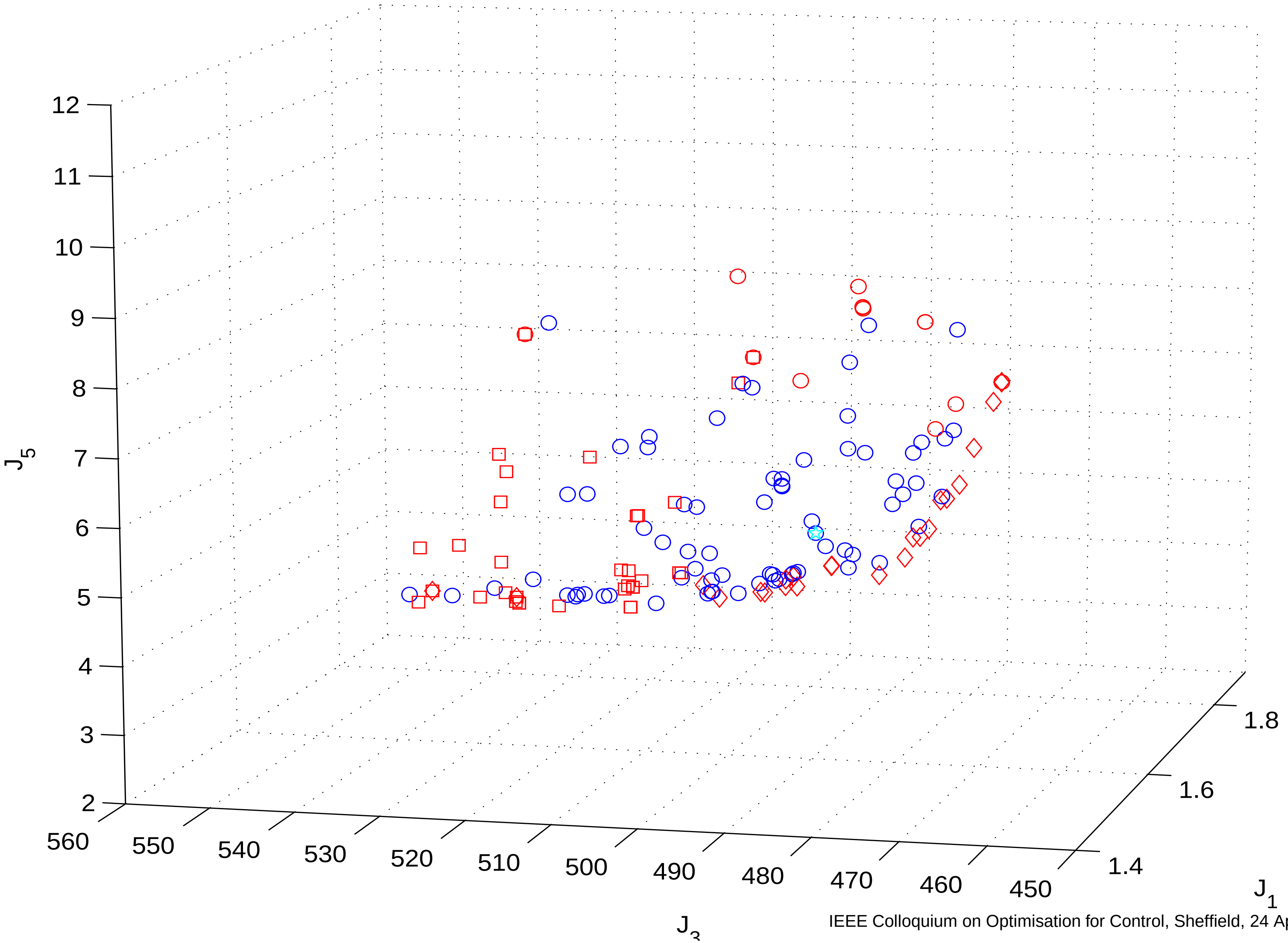
Three-objective optimisation



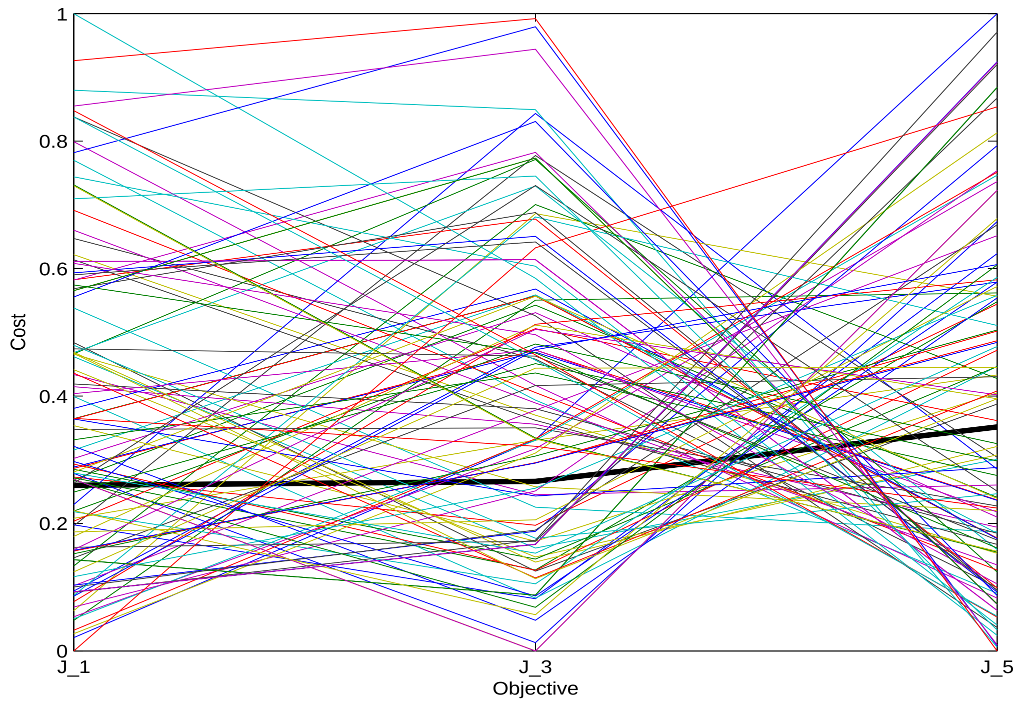
3D Pareto front

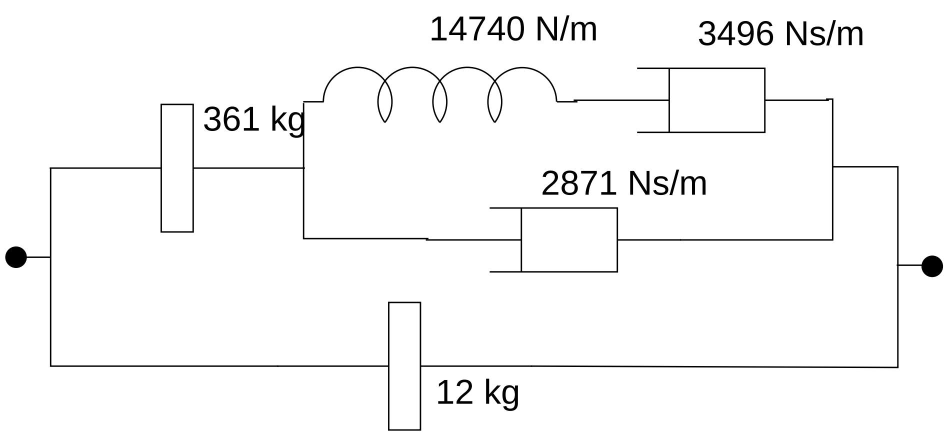


3D Pareto front



Realisation of the "best" suspension





Conclusions

- BMI optimisation over positive real controllers has been shown to be effective for two-objective optimisation problem. However, the design of a single controller requires several attempts.
- MOGA-base method- The Pareto-front can be investigated in a single optimisation run.
- More than two objectives can be included straightforwardly Thusm the designer engineer has a choice from among the Pareto-optimal set.
- A Possible disadvantage of using MOGA is that some experience may be needed to choose appropriate parameter values, such as the population size.

The end

Thank you,

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Reference:

A Molina-Cristobal, C Papageorgiou, G T Parks, M C Smith, P J Clarkson. Multi-objective Controller Design: Evolutionary Algorithms and Bilinear Matrix Inequalities for a Passive Suspension *Proceedings of the IFAC Workshop on Control Applications of Optimization, Cachan, France, April 2006*

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