The Scenario Approach: Robust Optimization and Application to Control

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A general fact:

- convex optimization is easy

but

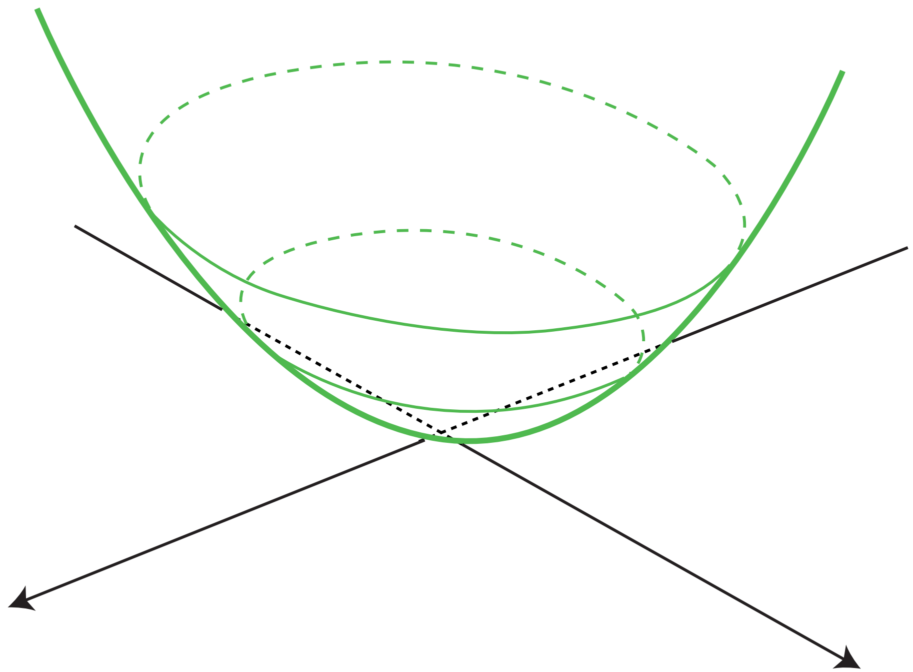
- robust convex optimization is hard

$$\begin{array}{l} \min \ c^T x \\ \text{subject to: } f(x,\delta) \le 0, \quad \forall \delta \in \Delta \end{array}$$

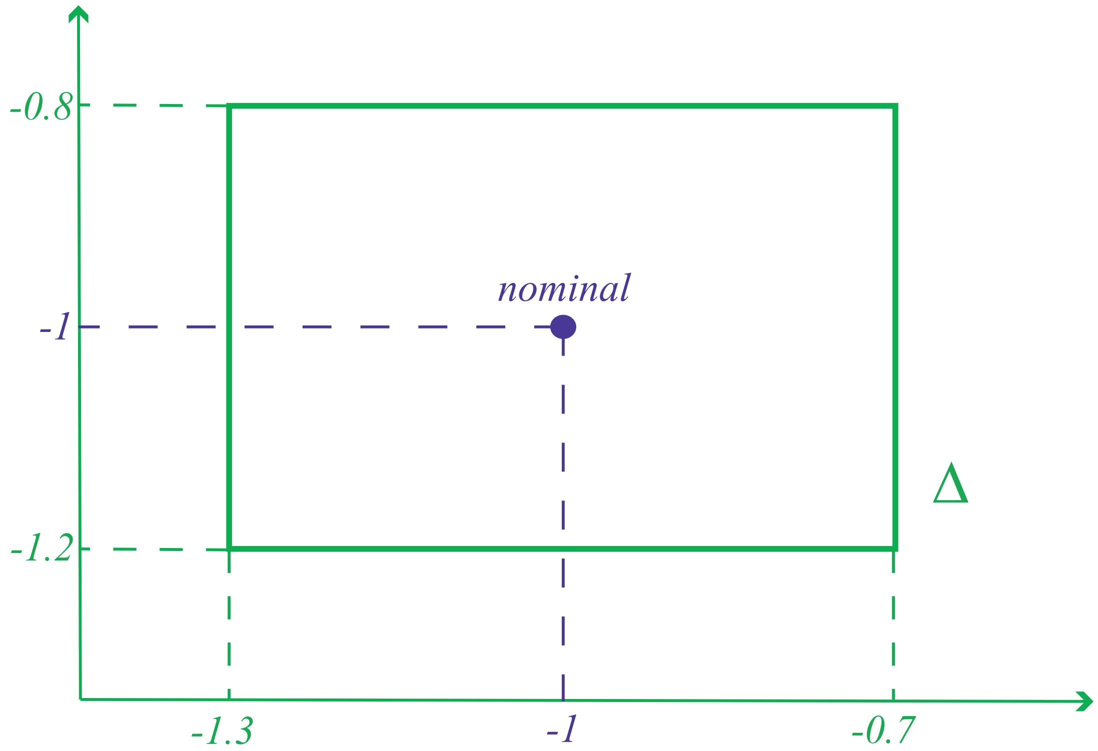
Example (stability)

$$\dot{x} = Ax$$

$$\begin{cases} P \succ 0 \\ A^T P + PA \prec 0 \qquad LMI - convex \end{cases}$$



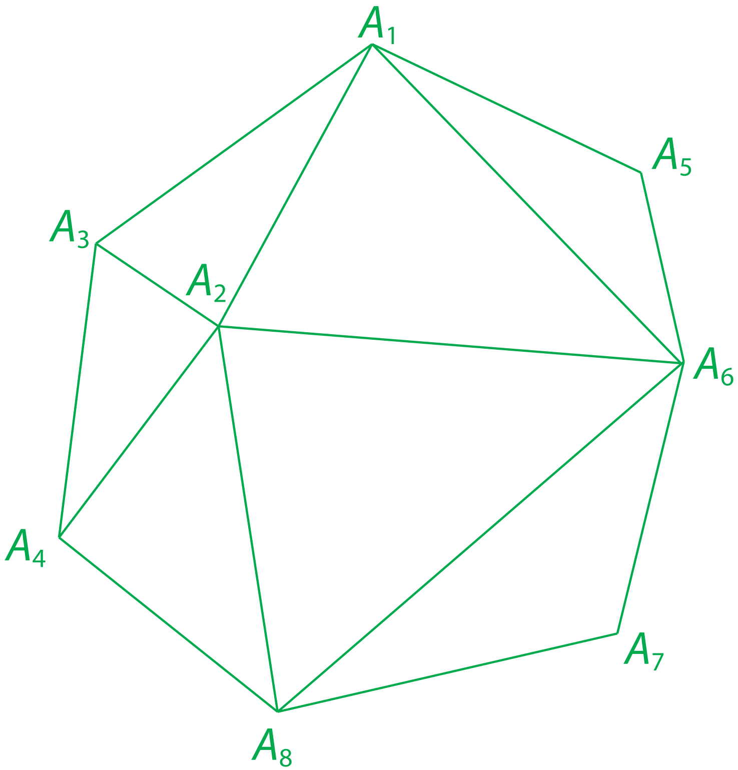
Uncertainty - robust



$$\dot{x} = A(\delta)x$$

$$\begin{cases} P \succ 0 \\ A(\delta)^T P + PA(\delta) \prec 0 \qquad \forall \delta \in \Delta \end{cases}$$

infinite number of constraints!!!

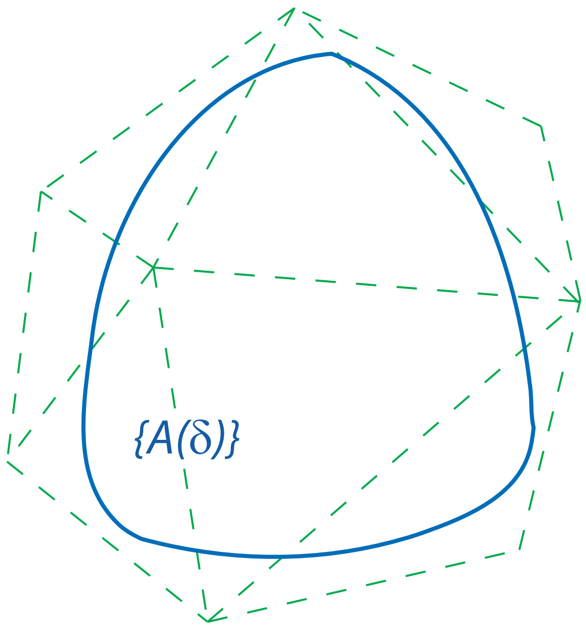


$$A(\delta) = \sum_i \delta_i A_i$$

(convex: $0 \leq \delta_i \leq 1 \;\; \sum_i \delta_i = 1$)

$$\begin{cases} P \succ 0 \\ A_1^T P + P A_1 \prec 0 \\ \vdots \\ A_n^T P + P A_n \prec 0 \end{cases}$$

Towards generality



relaxation

$$\begin{cases} P \succ 0 \\ A(\delta)^T P + P A(\delta) \prec 0 \end{cases}$$

QS - Quadratic Stability

$-P = P_0 + \delta_1 P_1 + \cdots + \delta_m P_m$ AQS - Affine Quadratic Stability

$-P(z, \delta)$ *linear in* z GQS - Generalized Quadratic Stability

$-P(\delta)$ general case

Other problems in control

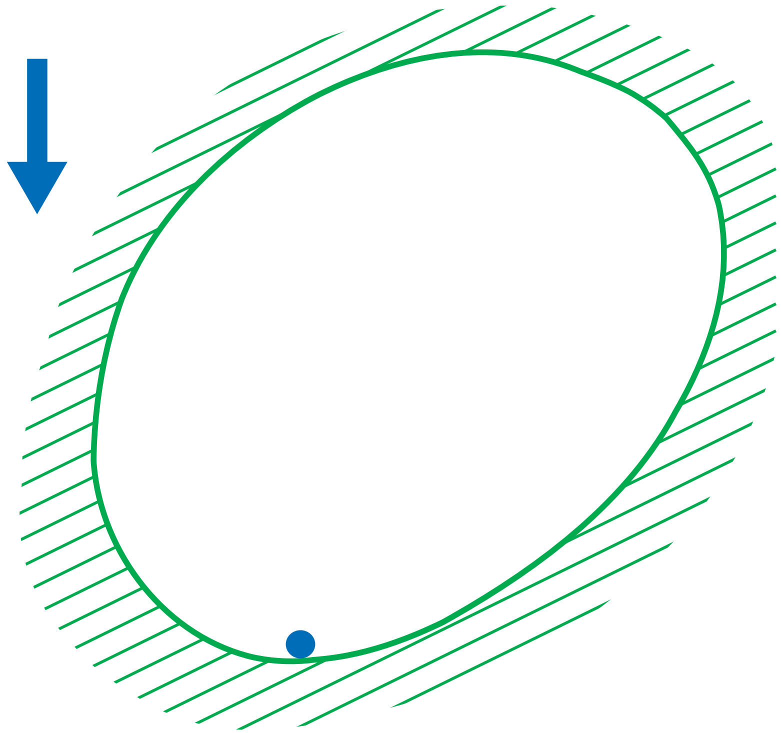
- state-feedback stabilization
- H_∞ control
- H_2 control
- LPV control

$$\vdots$$

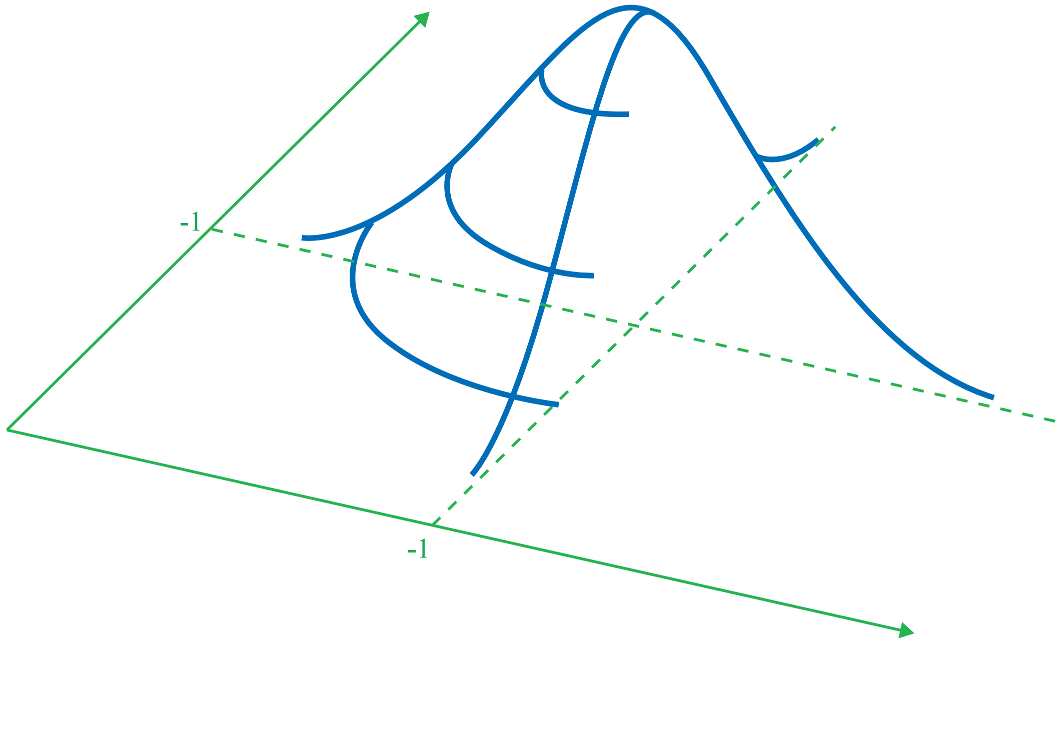
Robust Convex Optimization

$$\min \; c^T x$$

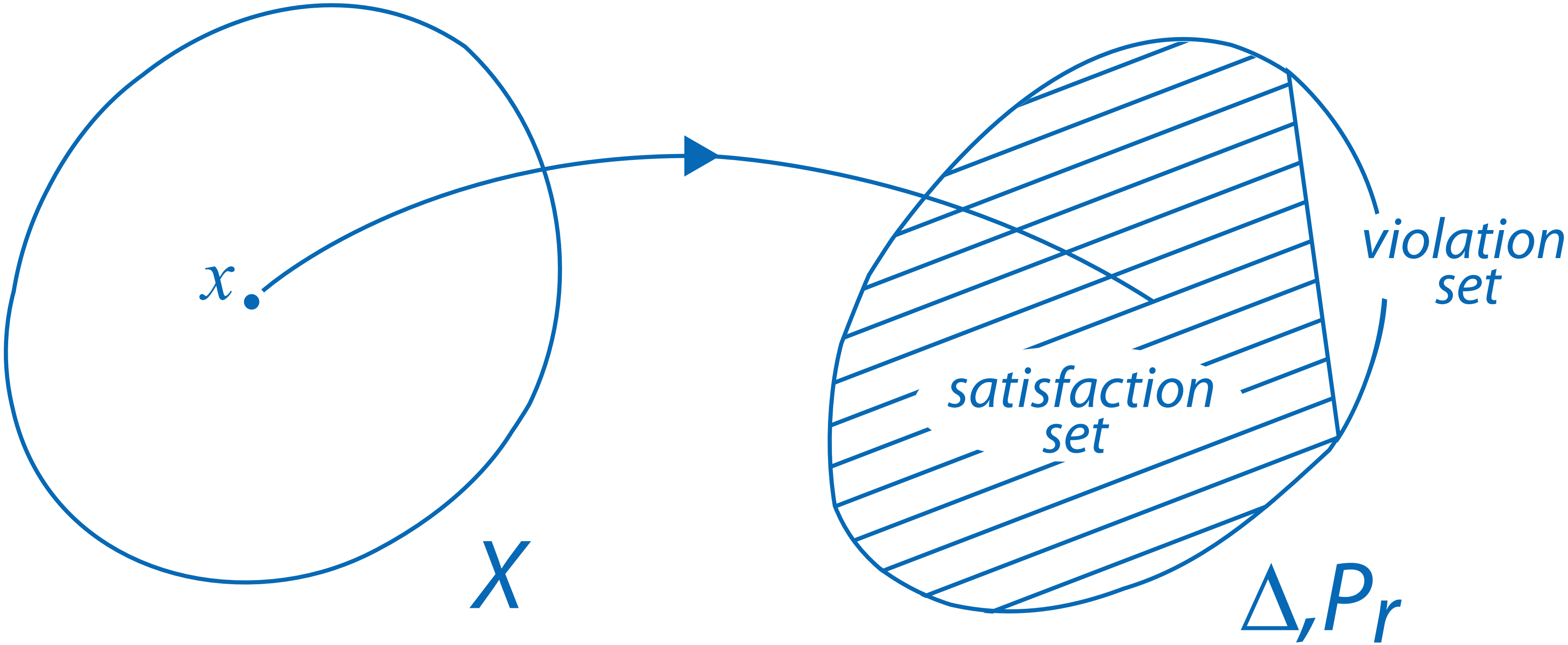
$$\text{subject to: } f(x, \delta) \leq 0, \quad \forall \delta \in \Delta$$



Uncertainty



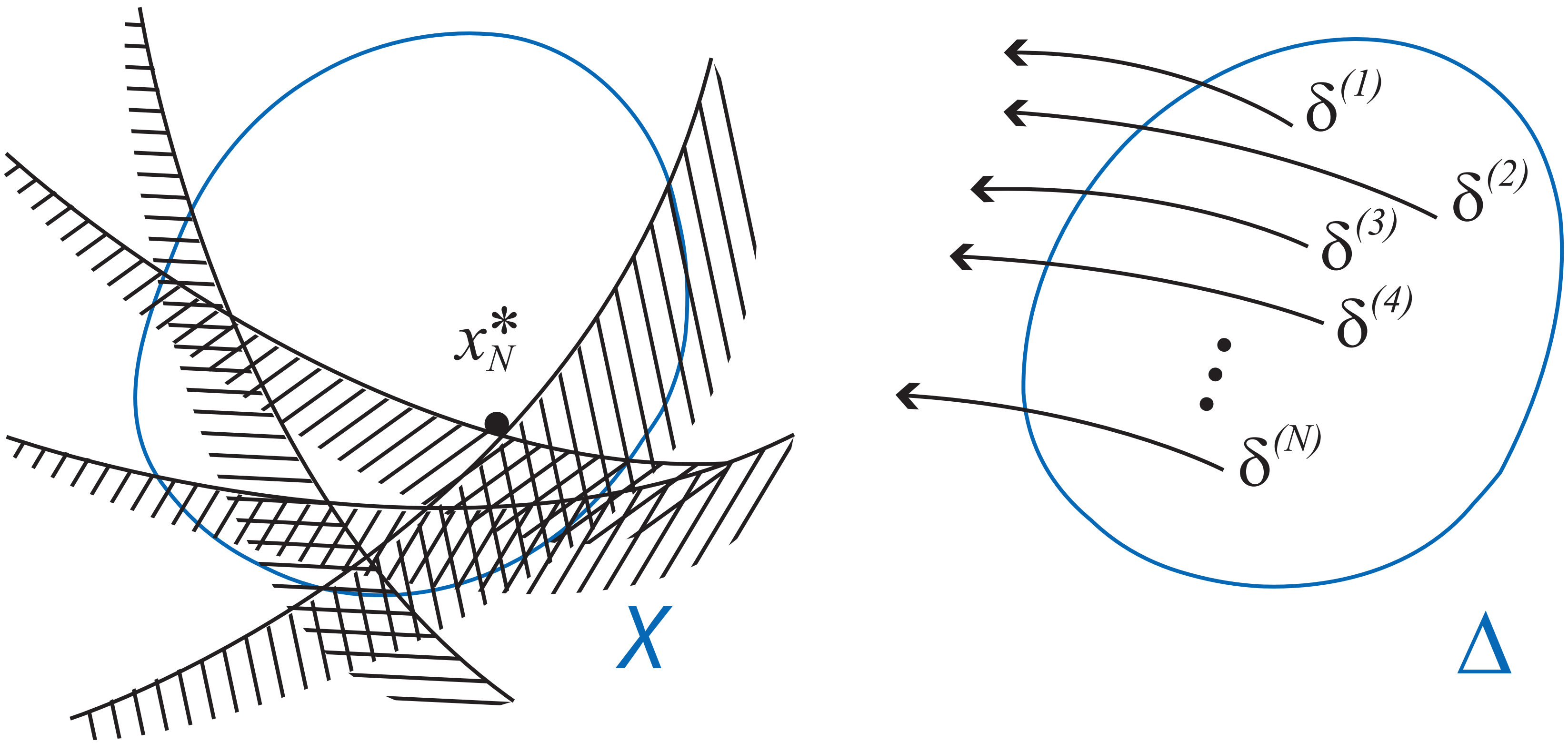
Violation set



$$Pr(\text{violation set}) \leq \epsilon$$

- chance-constrained optimization

The "Scenario" Paradigm



SCP_N = scenario convex program

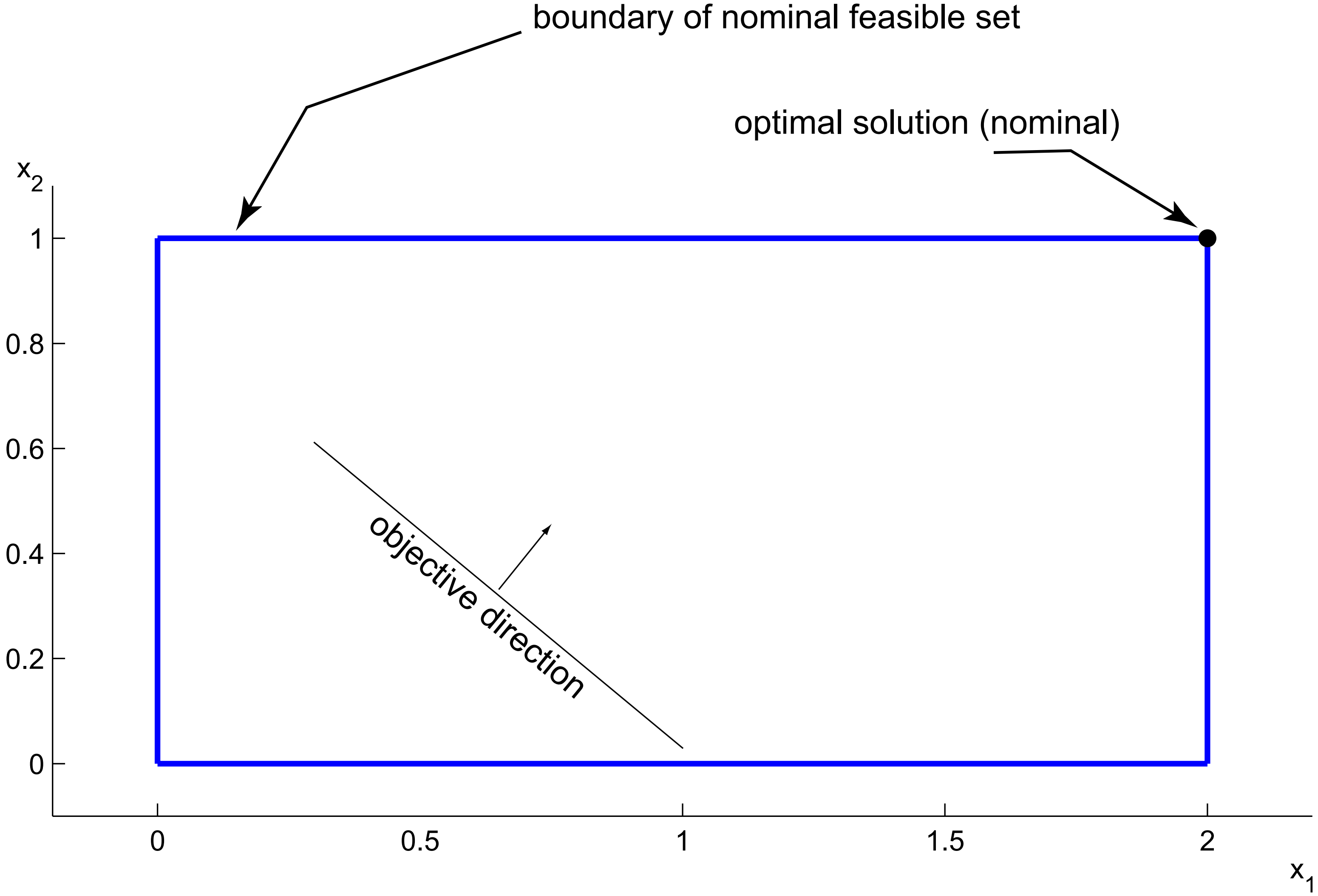
- SCP_N is a standard finite convex optimization problem
- x_N^* is superoptimal

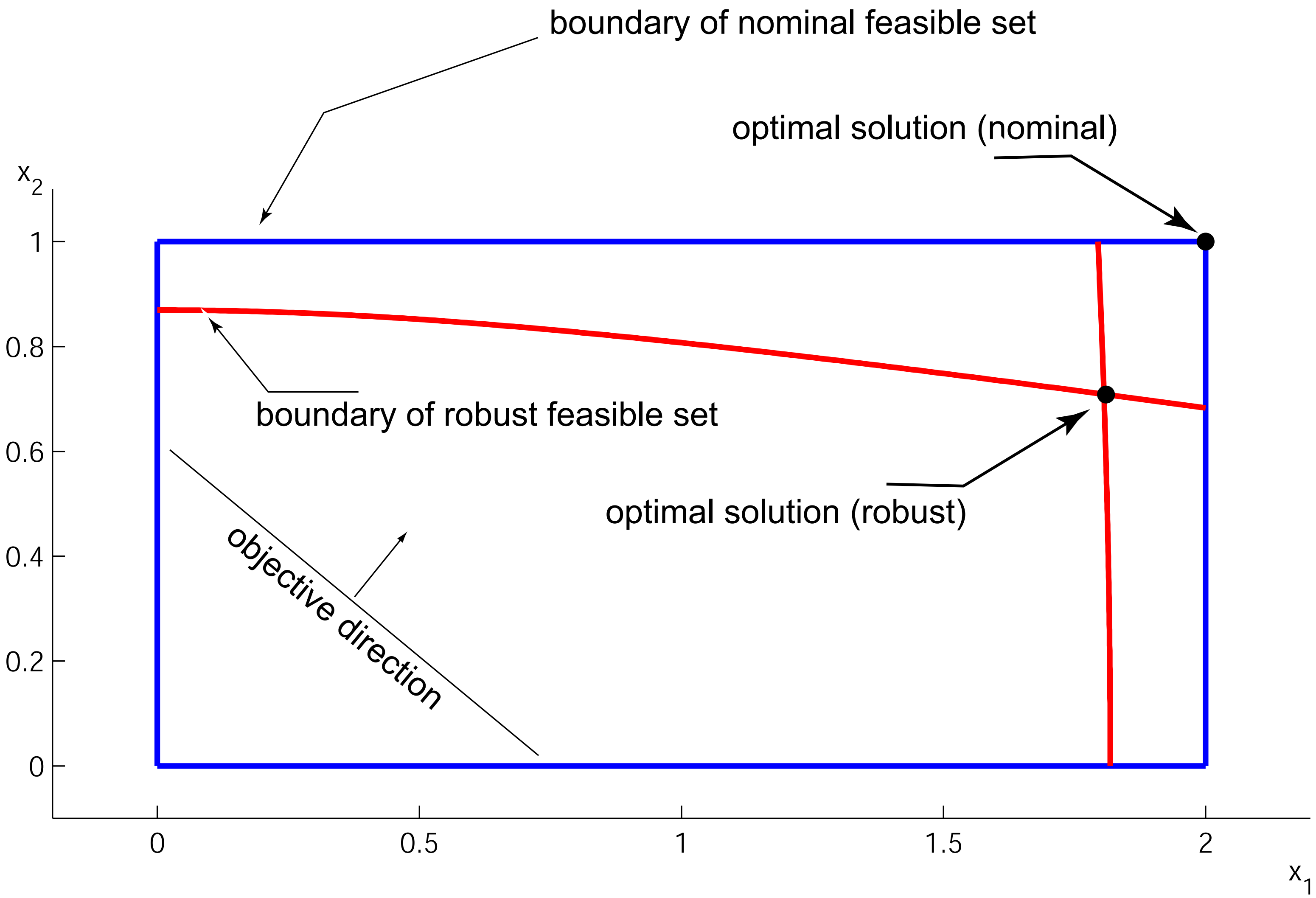
Fundamental question:

how feasible is x_N^*?

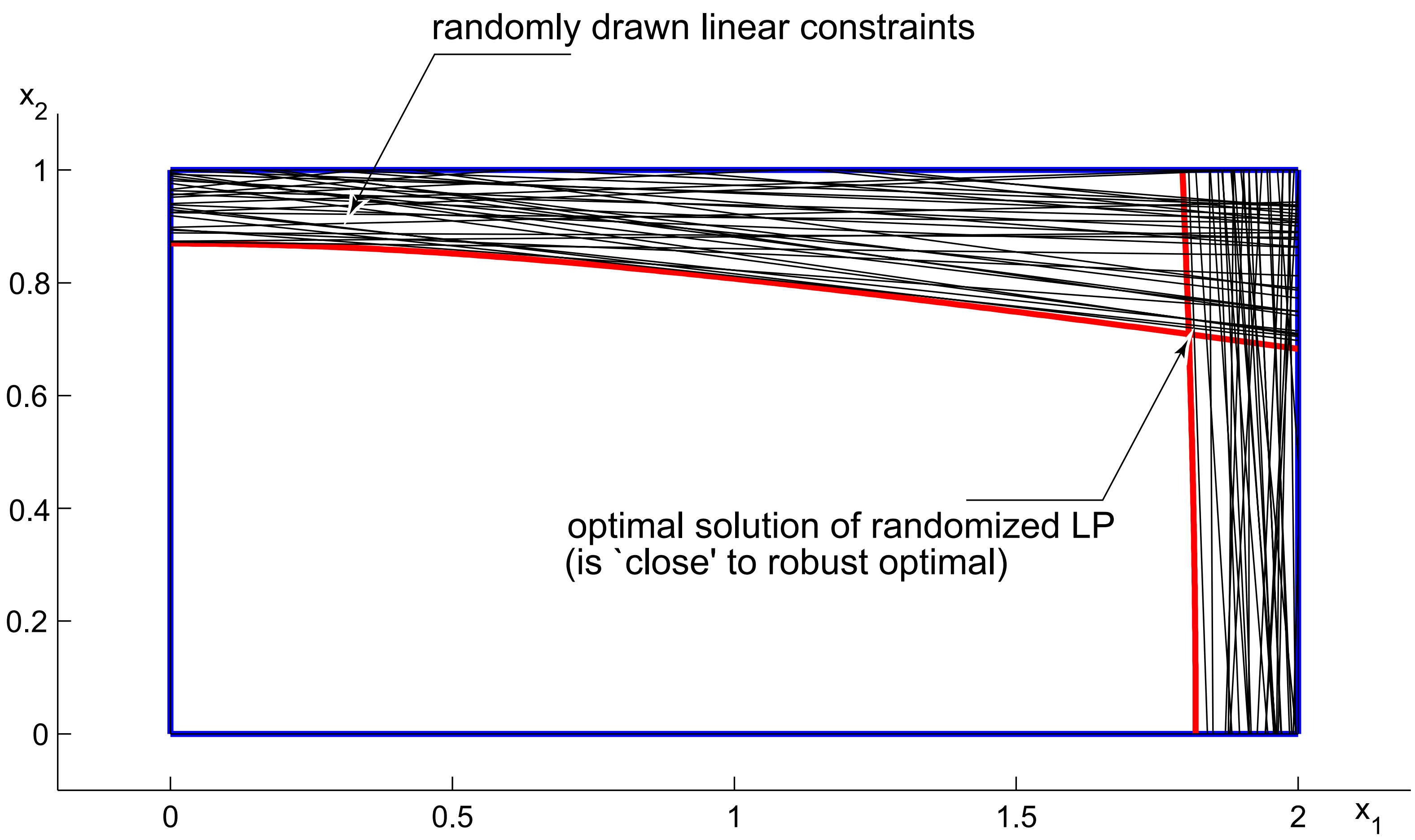
Example

$$\begin{array}{llll} \text{RLP:} & \min & c^T x & c^T = [-1 \quad -1] \\ & \text{subject to} & a_1^T x \le 2, & a_1^T = [1 \;\; 0] + \rho_1 \delta_1, \quad \rho_1 = 0.1, \; |\delta_1| \le 1 \\ & & a_2^T x \le 1, & a_2^T = [1 \;\; 0] + \rho_2 \delta_2, \quad \rho_2 = 0.15, \; |\delta_2| \le 1 \\ & & a_3^T x \le 0, & a_3^T = [-1 \;\; 0] \\ & & a_4^T x \le 0, & a_4^T = [0 \quad -1] \end{array}$$


boundary of nominal feasible set
optimal solution (nominal)
objective direction
x2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
x1



boundary of nominal feasible set
optimal solution (nominal)
boundary of robust feasible set
optimal solution (robust)
objective direction
x_2
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
x_1


randomly drawn linear constraints
optimal solution of randomized LP
(is `close' to robust optimal)
x2
1
0.8
0.6
0.4
0.2
0
0.5
1.5
2
x1

Fundamental question:

how feasible is x_N^*?

generalization $\Longrightarrow$ need for structure

Good news: the structure we need is convexity

- double role of convexity:
  - practice (computation)
  - theory (generalization)

Theorem

Fix $\epsilon \in (0,1)$ (violation parameter)

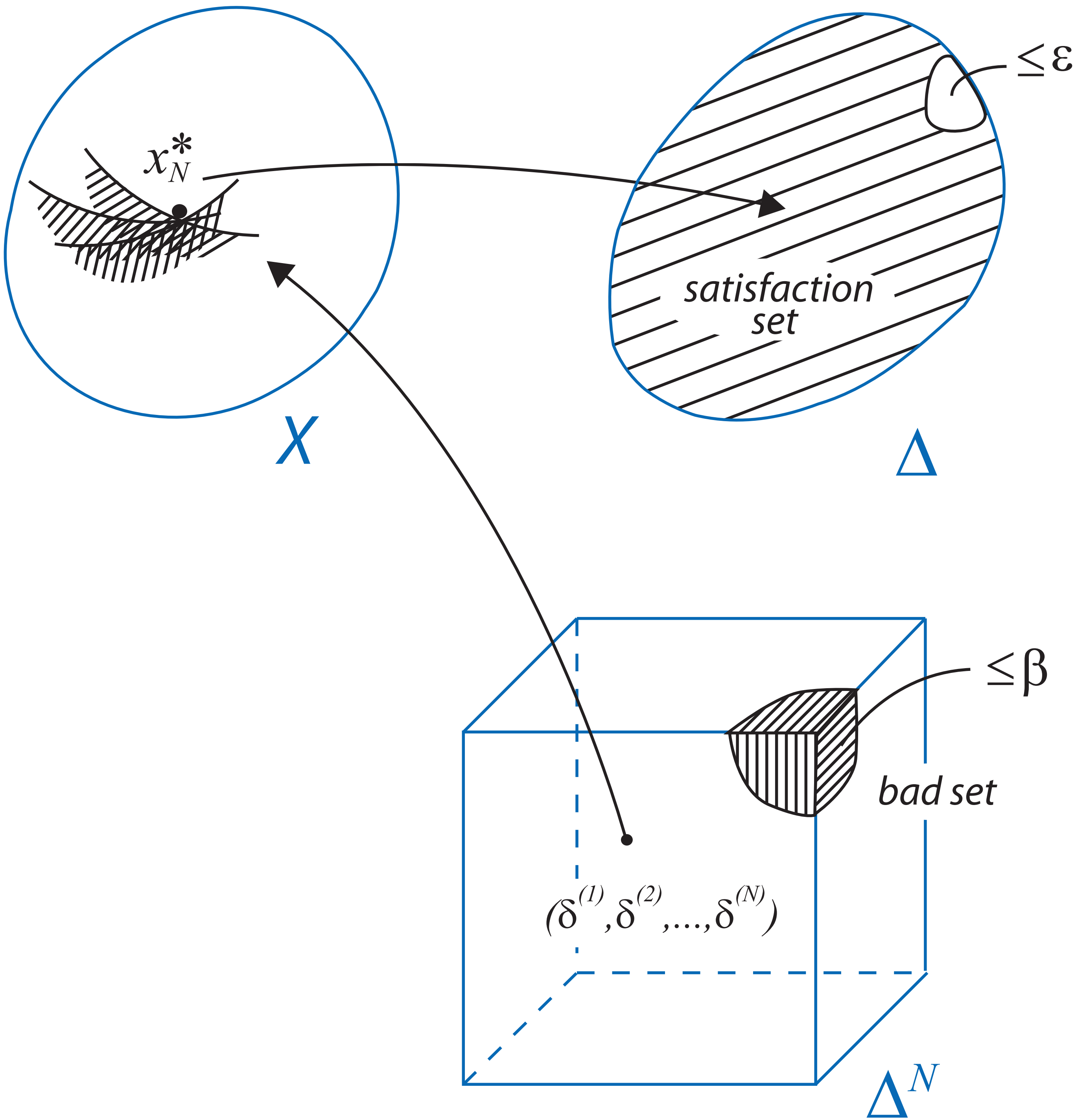
$\beta \in (0,1)$ (confidence parameter)

If $N \geq N(\epsilon,\beta) \doteq \frac{2}{\epsilon}\ln\frac{1}{\beta} + 2n_x + \frac{2n_x}{\epsilon}\ln\frac{2}{\epsilon}$,

then,

with probability $\geq 1-\beta$,

x_N^* is ϵ-level robustly feasible.



x_N^*
$\leq \varepsilon$
satisfaction set
X
Δ
$\leq \beta$
bad set
$(\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(N)})$
Δ^N

Extensions:

- SCP_N is unfeasible
- x_N^* is not unique
- SCP_N is feasible, but x_N^* does not exist

Comments:

$$N \geq \frac{2}{\epsilon} \ln \frac{1}{\beta} + 2n_x + \frac{2n_x}{\epsilon} \ln \frac{2}{\epsilon}$$

- N usually tractable by standard solvers
- N easy to compute
- N independent of Pr
- permits to address problems otherwhise intractable

$\underline{Ex}$: stability of $A(\delta)$

$P(z, \delta) \quad GQS$

- even when RCP is tractable, SCP_N gives a way to trade probability of violation for performance

$\rightarrow \epsilon =$ tuning knob

Example (stability-synthesis)

$$\dot{x} = \begin{bmatrix} 0.5\delta_2 & 1+\delta_1 \\ -(1+\delta_1)^2 & 2(0.1+0.5\delta_2)(1+\delta_1) \end{bmatrix} x + \begin{bmatrix} 10 \\ 15 \end{bmatrix} u$$

$|\delta_1| \leq 1, \quad |\delta_2| \leq 1$

Goal: design $u = Kx$ such that the closed-loop is quadratically stable

$$A_{cl}(\delta) = A(\delta) + BK$$

Lyapunov condition:

$$PA^T(\delta) + A(\delta)P + \underbrace{PK^T}_{=:Y^T} B^T + B \underbrace{KP}_{=:Y} \prec 0 \qquad \forall \delta \in \Delta$$

$$K = YP^{-1}$$

$$\min_{P,Y,\gamma} \quad \gamma$$

$$\text{subject to } -I \preceq \begin{bmatrix} -P & 0 \\ 0 & PA^T(\delta) + A(\delta)P + Y^T B^T + BY \end{bmatrix} \preceq \gamma I, \quad \forall \delta$$

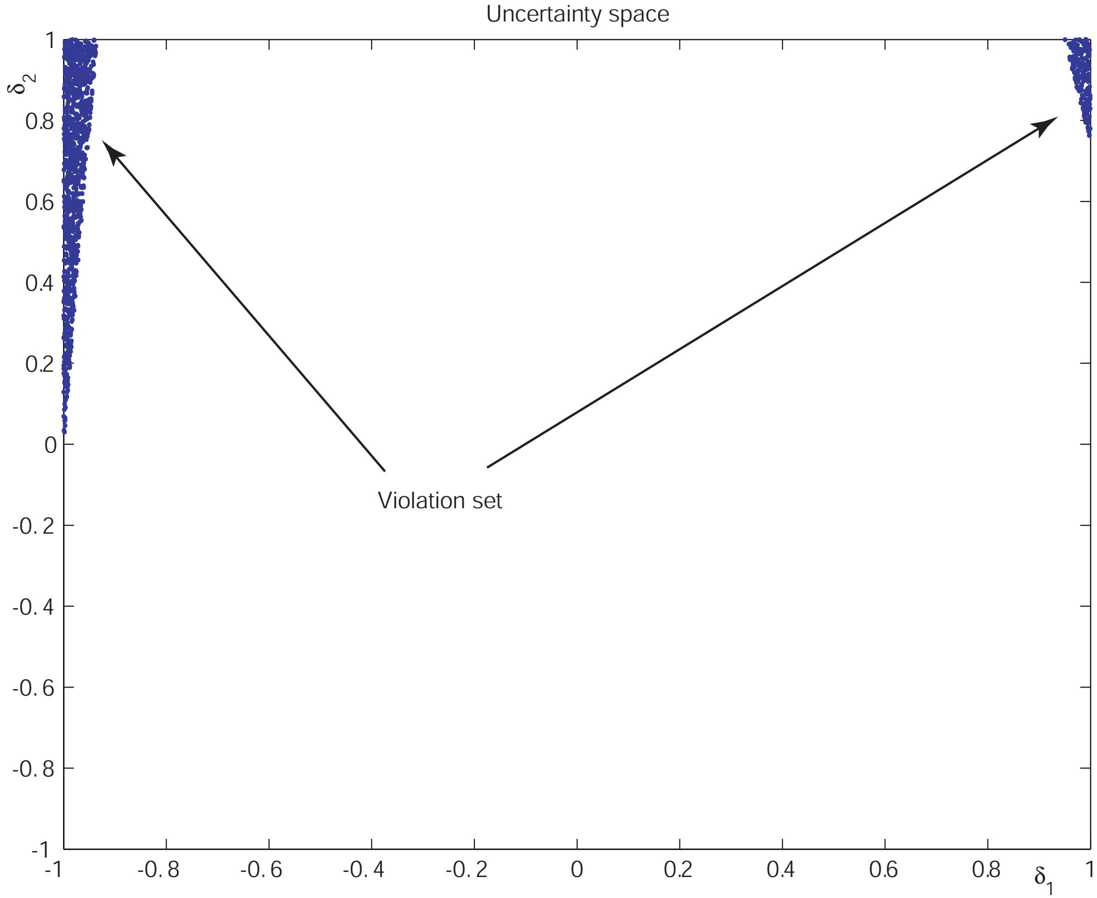
$$\left.\begin{array}{rcl}\epsilon & = & 0.05 \\ \beta & = & 0.001\end{array}\right\} \rightarrow N = 1174$$

$$P = \left[\begin{array}{cc} 0.0273 & -0.0212 \\ -0.0212 & 0.4852 \end{array}\right] \quad Y = \left[\begin{array}{cc} -0.1620 & -0.2280 \end{array}\right]$$

$$K = [-6.5162 \quad -0.7550]$$

$$\gamma^* < 0$$

A-posteriori: Monte-Carlo analysis

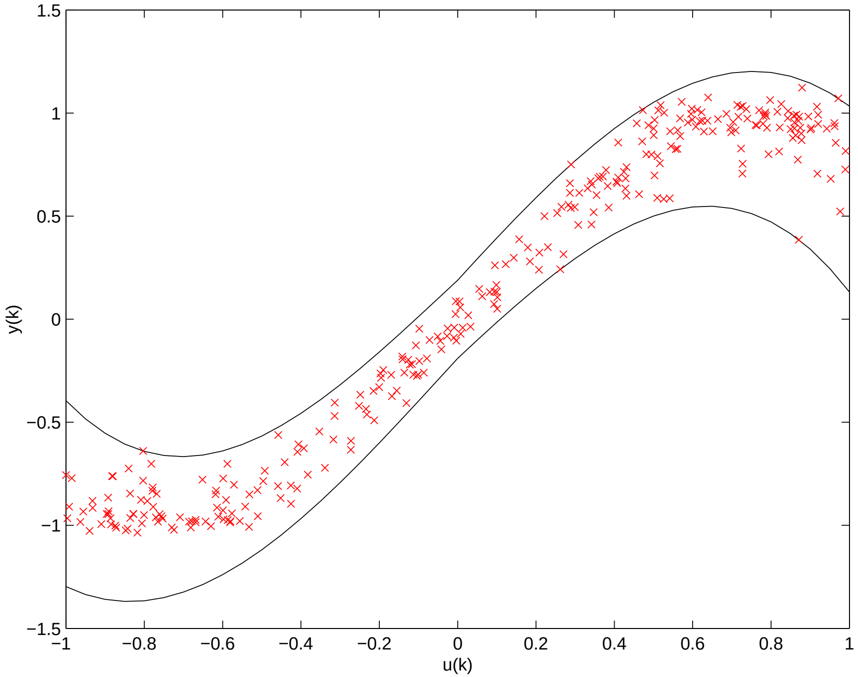


$N = 100,000$

$\hat{\epsilon} = 0.0096$

Other problems in systems theory

- construction of interval models for prediction



- min-max identification

$$\min_{M} \max_{S} d(S, M) \quad (e.g.\ d(S, M) = E[(y - \hat{y})^2])$$

Conclusions

- Finite convex optimization is simple, but semi-infinite convex optimization is hard in general

- The scenario approach offers a viable way to solve semi-infinite convex optimization problems in a risk-adjusted sense, based on a generalization result valid for all convex problems

- ϵ trades robustness for performance

References

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