MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 13 — Solutions

Please hand in Exercises 2 and 3 by the Wednesday lecture of Week 7. I would prefer paper, but if that is not possible for some reason, then you can send me a scan by email.

Throughout this problem sheet we use the notation of Section 18 of the notes. In particular, we use the elements $\theta_n \in C_n(\Delta_n)$ and $\kappa_n \in C_{n+1}(\Delta_n)$ that are defined in that section.

Exercise 1. Example 18.5 gives the following formula:

$$\theta_2 = \langle e_{012}, e_{12}, e_2\rangle - \langle e_{012}, e_{12}, e_1\rangle - \langle e_{012}, e_{02}, e_2\rangle + \langle e_{012}, e_{02}, e_0\rangle + \langle e_{012}, e_{01}, e_1\rangle - \langle e_{012}, e_{01}, e_0\rangle.$$

We could write this with abbreviated notation as follows:

$$\theta_2 = \langle 012, 12, 2\rangle - \langle 012, 12, 1\rangle - \langle 012, 02, 2\rangle + \langle 012, 02, 0\rangle + \langle 012, 01, 1\rangle - \langle 012, 01, 0\rangle.$$

Use the same method and the same abbreviated notation to give a formula for θ_3 (which should have 24 terms).

Solution:

$$\begin{aligned}
(\delta_0)_*(\theta_2) =& \langle 123, 23, 3\rangle - \langle 123, 23, 2\rangle - \langle 123, 13, 3\rangle + \langle 123, 13, 1\rangle + \langle 123, 12, 2\rangle - \langle 123, 12, 1\rangle \\
(\delta_1)_*(\theta_2) =& \langle 023, 23, 3\rangle - \langle 023, 23, 2\rangle - \langle 023, 03, 3\rangle + \langle 023, 03, 0\rangle + \langle 023, 02, 2\rangle - \langle 023, 02, 0\rangle \\
(\delta_2)_*(\theta_2) =& \langle 013, 13, 3\rangle - \langle 013, 13, 1\rangle - \langle 013, 03, 3\rangle + \langle 013, 03, 0\rangle + \langle 013, 01, 1\rangle - \langle 013, 01, 0\rangle \\
(\delta_3)_*(\theta_2) =& \langle 012, 12, 2\rangle - \langle 012, 12, 1\rangle - \langle 012, 02, 2\rangle + \langle 012, 02, 0\rangle + \langle 012, 01, 1\rangle - \langle 012, 01, 0\rangle \\
\theta_3' =& \langle 123, 23, 3\rangle - \langle 123, 23, 2\rangle - \langle 123, 13, 3\rangle + \langle 123, 13, 1\rangle + \langle 123, 12, 2\rangle - \langle 123, 12, 1\rangle - \\
& \langle 023, 23, 3\rangle + \langle 023, 23, 2\rangle + \langle 023, 03, 3\rangle - \langle 023, 03, 0\rangle - \langle 023, 02, 2\rangle + \langle 023, 02, 0\rangle + \\
& \langle 013, 13, 3\rangle - \langle 013, 13, 1\rangle - \langle 013, 03, 3\rangle + \langle 013, 03, 0\rangle + \langle 013, 01, 1\rangle - \langle 013, 01, 0\rangle - \\
& \langle 012, 12, 2\rangle + \langle 012, 12, 1\rangle + \langle 012, 02, 2\rangle - \langle 012, 02, 0\rangle - \langle 012, 01, 1\rangle + \langle 012, 01, 0\rangle \\
\theta_3 =& \langle 0123, 123, 23, 3\rangle - \langle 0123, 123, 23, 2\rangle - \langle 0123, 123, 13, 3\rangle + \langle 0123, 123, 13, 1\rangle + \langle 0123, 123, 12, 2\rangle - \langle 0123, 123, 12, 1\rangle - \\
& \langle 0123, 023, 23, 3\rangle + \langle 0123, 023, 23, 2\rangle + \langle 0123, 023, 03, 3\rangle - \langle 0123, 023, 03, 0\rangle - \langle 0123, 023, 02, 2\rangle + \langle 0123, 023, 02, 0\rangle + \\
& \langle 0123, 013, 13, 3\rangle - \langle 0123, 013, 13, 1\rangle - \langle 0123, 013, 03, 3\rangle + \langle 0123, 013, 03, 0\rangle + \langle 0123, 013, 01, 1\rangle - \langle 0123, 013, 01, 0\rangle - \\
& \langle 0123, 012, 12, 2\rangle + \langle 0123, 012, 12, 1\rangle + \langle 0123, 012, 02, 2\rangle - \langle 0123, 012, 02, 0\rangle - \langle 0123, 012, 01, 1\rangle + \langle 0123, 012, 01, 0\rangle
\end{aligned}$$

Exercise 2. Give formulae for κ_1 (with 3 terms) and κ_2 (with 16 terms). Use abbreviated notation as in the previous exercise.

Solution:

$$\begin{aligned}
\kappa_0 =& 0 \\
\kappa_1' =& \iota_1 - \theta_1 = \langle 0, 1\rangle - \langle 01, 1\rangle + \langle 01, 0\rangle \\
\kappa_1 =& \beta(\kappa_1') = \langle 01, 0, 1\rangle - \langle 01, 01, 1\rangle + \langle 01, 01, 0\rangle \\
\iota_2 - \theta_2 =& \langle 0, 1, 2\rangle - \langle 012, 12, 2\rangle + \langle 012, 12, 1\rangle + \langle 012, 02, 2\rangle - \langle 012, 02, 0\rangle - \langle 012, 01, 1\rangle + \langle 012, 01, 0\rangle \\
(\delta_0)_*(\kappa_1) =& \langle 12, 1, 2\rangle - \langle 12, 12, 2\rangle + \langle 12, 12, 1\rangle \\
(\delta_1)_*(\kappa_1) =& \langle 02, 0, 2\rangle - \langle 02, 02, 2\rangle + \langle 02, 02, 0\rangle \\
(\delta_2)_*(\kappa_1) =& \langle 01, 0, 1\rangle - \langle 01, 01, 1\rangle + \langle 01, 01, 0\rangle \\
\sum_i (-1)^i (\delta_i)_*(\kappa_1) =& \langle 12, 1, 2\rangle - \langle 12, 12, 2\rangle + \langle 12, 12, 1\rangle - \\
& \langle 02, 0, 2\rangle + \langle 02, 02, 2\rangle - \langle 02, 02, 0\rangle + \\
& \langle 01, 0, 1\rangle - \langle 01, 01, 1\rangle + \langle 01, 01, 0\rangle
\end{aligned}$$

By applying β to the terms above we get

$$\begin{aligned}\kappa_2 =&\beta(\kappa_2')\\ =&\langle 012,0,1,2\rangle -\\ &\langle 012,012,12,2\rangle + \langle 012,012,12,1\rangle + \langle 012,012,02,2\rangle -\\ &\langle 012,012,02,0\rangle - \langle 012,012,01,1\rangle + \langle 012,012,01,0\rangle -\\ &\langle 012,12,1,2\rangle + \langle 012,12,12,2\rangle - \langle 012,12,12,1\rangle +\\ &\langle 012,02,0,2\rangle - \langle 012,02,02,2\rangle + \langle 012,02,02,0\rangle -\\ &\langle 012,01,0,1\rangle + \langle 012,01,01,1\rangle - \langle 012,01,01,0\rangle\end{aligned}$$

Exercise 3. We define slightly modified versions of sd and σ as follows. Define $\mathrm{sd}_0' \colon C_0(X) \to C_0(X)$ to be the identity, and define $\sigma_0' \colon C_0(X) \to C_1(X)$ to be zero. Define $\lambda, \rho \colon \Delta_1 \to \Delta_1$ and $\phi \colon \Delta_2 \to \Delta_1$ by

$$\begin{aligned}\lambda(t_0,t_1) &= (t_0 + t_1/2, t_1/2)\\ \rho(t_0,t_1) &= (t_0/2, t_0/2 + t_1)\\ \phi(t_0,t_1,t_2) &= (t_0 + t_1/2, t_1/2 + t_2).\end{aligned}$$

For $u \colon \Delta_1 \to X$ put $\mathrm{sd}_1'(u) = u \circ \lambda + u \circ \rho \in C_1(X)$ and $\sigma_1'(u) = -(u \circ \phi) \in C_2(X)$. Extend this linearly to define $\mathrm{sd}_1' \colon C_1(X) \to C_1(X)$ and $\sigma_1' \colon C_1(X) \to C_2(X)$.

(a) Check that $\partial(\sigma_1'(u)) + \sigma_0'(\partial(u)) = u - \mathrm{sd}_1'(u)$.

(b) What can you say about the relationship between sd_1' and sd_1?

Note: Here X is an arbitrary space, which may not have anything to do with $\mathbb{R}^N$. Even if $X = \mathbb{R}^N$, the map $u \colon \Delta_1 \to X$ need not be linear. Thus, you should not be using ideas or notation that are only valid for linear simplices in $\mathbb{R}^N$.

Solution:

(a) Here we have $\sigma_0' = 0$ so we need only consider $\partial(\sigma_1'(u))$. Here $\sigma_1'(u) = -(u \circ \phi)$ so

$$\partial(\sigma_1'(u)) = -u \circ \phi \circ \delta_0 + u \circ \phi \circ \delta_1 - u \circ \phi \circ \delta_2.$$

We also have

$$\begin{aligned}(\phi \circ \delta_0)(t_0,t_1) &= \phi(0,t_0,t_1) = (t_0/2, t_0/2 + t_1) = \rho(t_0,t_1)\\ (\phi \circ \delta_1)(t_0,t_1) &= \phi(t_0,0,t_1) = (t_0,t_1)\\ (\phi \circ \delta_2)(t_0,t_1) &= \phi(t_0,t_1,0) = (t_0 + t_1/2, t_1/2) = \lambda(t_0,t_1),\end{aligned}$$

so

$$\partial(\sigma_0'(u)) = -u \circ \rho + u - u \circ \lambda = u - \mathrm{sd}_1'(u)$$

as required.

(b) For a path u, we have

$$\begin{aligned}\mathrm{sd}_1'(u) &= \text{ (first half of } u) \;+\; (\text{second half of } u)\\ \mathrm{sd}_1(u) &= -\text{ (first half of } u\text{, reversed)} \;+\; (\text{second half of } u)\end{aligned}$$