MAS61015 ALGEBRAIC TOPOLOGY — PROBLEM SHEET 10

Please hand in Exercises 2 and 3 by the Wednesday lecture of Week 4. I would prefer paper, but if that is not possible for some reason, then you can send me a scan by email.

Exercise 1. Let U be an abelian group. Consider the chain complex

$$A_* = (U \xleftarrow{0} U \xleftarrow{1} U \xleftarrow{0} U \xleftarrow{1} U \xleftarrow{\cdots})$$

(with the first group in degree zero).

(a) What is H_*A?
(b) Define $f\colon A_* \to A_*$ by $f_0 = 1$ and $f_k = 0$ for all $k \neq 0$. Prove that f is chain-homotopic to the identity.

Exercise 2. Consider the chain complex U_* where $U_n = \mathbb{Z}/100$ and $d_n(a) = 10a$ for all $n \in \mathbb{Z}$. Prove that $H_*(U) = 0$ but that that the identity map $\mathrm{id}\colon U_* \to U_*$ is not chain homotopic to zero.

Exercise 3. Let U_* be a chain complex in which all the groups U_k are finite-dimensional vector spaces over $\mathbb{Q}$, and all the differentials $d\colon U_k \to U_{k-1}$ are $\mathbb{Q}$-linear.

(a) For each n, choose a basis $b_{n,1}, \dots, b_{n,p(n)}$ for $B_n(U)$.
(b) Explain why we can choose elements $v_{n+1,1}, \dots, v_{n+1,p(n)} \in U_{n+1}$ such that $d(v_{n+1,k}) = b_{n,k}$ for all k.
(c) Explain why we can choose additional elements $h_{n,1}, \dots, h_{n,q(n)} \in Z_n(U)$ such that $b_{n,1}, \dots, b_{n,p(n)}, h_{n,1}, \dots, h_{n,q(n)}$ is a basis for $Z_n(U)$. Describe $H_n(U)$ in terms of this basis.
(d) Explain why the list $v_{n,1}, \dots, v_{n,p(n-1)}, b_{n,1}, \dots, b_{n,p(n)}, h_{n,1}, \dots, h_{n,q(n)}$ is a basis for U_n.
(e) Put $V_n = \mathrm{span}(h_{n,1}, \dots, h_{n,q(n)})$, and consider this as a chain complex with $d = 0$. Construct an injective chain map $i\colon V_* \to U_*$ and a surjective chain map $r\colon U_* \to V_*$.
(f) Define $s\colon U_n \to U_{n+1}$ by $s(b_{n,i}) = v_{n+1,i}$ and $s(v_{n,i}) = 0$ and $s(h_{n,i}) = 0$. Use this to show that U_* is chain homotopy equivalent to V_*.

Exercise 4. Consider the following matrices:

$$D = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplication by D gives a homomorphism $\mathbb{Z}^3 \to \mathbb{Z}^3$, and similarly for the other three matrices.

(a) Show that the sequence

$$A_* = (\mathbb{Z}^3 \xleftarrow{D} \mathbb{Z}^3 \xleftarrow{T} \mathbb{Z}^3 \xleftarrow{D} \mathbb{Z}^3 \xleftarrow{T} \mathbb{Z}^3 \leftarrow \cdots)$$

is a chain complex.
(b) Find $UT + DV$ and $TU + VD$.
(c) Use (b) to construct a chain homotopy between certain maps $A_* \to A_*$.
(d) Use this to calculate H_*A.