Algebraic Topology Exam Questions

This is a collection of questions taken from Algebraic Topology exams over a number of years. To some extent, they have been modified to be compatible with the version of the course taught in 2021-22, but some differences remain.

1 Compactness

Some questions in this section use ideas that are specific to metric spaces rather than general topological spaces. These ideas are not developed in the current version of the course.

(1) Let X be a metric space.

(a) Let Y be a compact subspace of X. Prove that Y is closed in X.

(b) Let Y and Z be two compact subspaces of X. Prove that $Y \cup Z$ is compact.

(c) Deduce (or prove otherwise) that every finite space is compact.

(d) Let Y and Z be compact metric spaces. Prove that $Y \times Z$ is compact.

(e) Conversely, let Y and Z be metric spaces such that $Z \neq \emptyset$ and $Y \times Z$ is compact. Prove that Y is compact.

(f) Put $X = \{(x,y,z) \in \mathbb{R}^3 \ : \ x^4 + y^4 + z^4 = 1\}$. Prove that X is compact. You may use general theorems provided that you state them precisely.

Solution:

(a) Let (y_n) be a sequence in Y, converging to some point $x \in X$. Clearly any subsequence converges to x also. By compactness, some subsequence (y_{n_k}) converges to some $y \in Y$, and as limits are unique we must have $x = y$, so $x \in Y$. This means that Y is closed, as required.

(b) Let (x_n) be a sequence in $Y \cup Z$. Then either $x_n \in Y$ for infinitely many n, or $x_n \in Z$ for infinitely many n. In the first case, we can choose a subsequence (x'_n) of (x_n) such that $x'_n \in Y$ for all n in other words we have a sequence in Y. As Y is compact, some subsequence (x''_n) of (x'_n) converges in Y, and thus in $Y \cup Z$. The other case is similar, so in either case some subsequence of (x_n) converges in $Y \cup Z$. This implies that $Y \cup Z$ is compact.

(c) If X has only one point then every sequence converges so X is compact. If X has $n > 1$ points, we can write it in the form $X = Y \cup Z$ where $|Y| = n - 1$ and $|Z| = 1$, so Y and Z are compact by induction, so X is compact by (ii).

(d) Let (w_n) be a sequence in $Y \times Z$, with $w_n = (y_n, z_n)$ say. As Y is compact, some subsequence (y_{n_k}) converges to some $y \in Y$. Put $y'_k = y_{n_k}$ and $z'_k = z_{n_k}$ and $w'_k = (y'_k, z'_k) = w_{n_k}$. As Z is compact, some subsequence z'_{k_j} converges to some point $z \in Z$. Put $y''_j = y'_{k_j}$ and $z''_j = z'_{k_j}$ and $w''_j = (y''_j, z''_j) = w'_{k_j}$. As (y''_j) is a subsequence of the sequence (y'_k) which converges to y, we see that $y''_j \to y$. By assumption we have $z''_j \to z$, so $w''_j \to (y,z)$. Thus, some subsequence of (w_n) converges in $Y \times Z$, proving that $Y \times Z$ is compact as claimed.

(e) As $Z \neq \emptyset$ we can choose a point $a \in Z$. Let $p \colon Y \times Z \to Y$ be defined by $p(y,z) = y$. We have $p(y,a) = y$, which shows that p is surjective. In general, if $f \colon A \to B$ is a surjective continuous map of spaces and A is compact we know that B is compact. As $Y \times Z$ is assumed compact, we deduce that Y is compact.

(e) If $(x,y,z) \in X$ then $x^4 \leq x^4 + y^4 + z^4 = 1$ so $|x| \leq 1$. Similarly, we see that $|y| \leq 1$ and $|z| \leq 1$, which implies that X is bounded. I claim that it is also closed in $\mathbb{R}^3$. Indeed, suppose we have a sequence $a_n = (x_n, y_n, z_n)$ in X converging to some point $a = (x,y,z) \in \mathbb{R}^3$. then $x_n^4 + y_n^4 + z_n^4 = 1$ and $x_n \to x$, $y_n \to y$ and $z_n \to z$, so by the algebra of limits we have

$$x^4 + y^4 + z^4 = \lim(x_n^4 + y_n^4 + z_n^4) = 1,$$

so $a \in X$.

A bounded closed subset of $\mathbb{R}^n$ is compact, so we deduce that X is compact as claimed.

(2) Let X be a metric space.

(a) Let Y be a compact subspace of X. Prove that Y is closed in X.

(b) Let Y and Z be two compact subspaces of X. Prove that $Y \cup Z$ is compact.

(c) Deduce (or prove otherwise) that every finite space is compact.

(d) Let Y and Z be compact metric spaces. Prove that $Y \times Z$ is compact.

(e) Conversely, let Y and Z be metric spaces such that $Z \neq \emptyset$ and $Y \times Z$ is compact. Prove that Y is compact.

(f) Put $X = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^4 + y^4 + z^4 = 1\}$. Prove that X is compact. You may use general theorems provided that you state them precisely.

Solution:

(a) Let (y_n) be a sequence in Y, converging to some point $x \in X$. Clearly any subsequence converges to x also. By compactness, some subsequence (y_{n_k}) converges to some $y \in Y$, and as limits are unique we must have $x = y$, so $x \in Y$. This means that Y is closed, as required.

(b) Let (x_n) be a sequence in $Y \cup Z$. Then either $x_n \in Y$ for infinitely many n, or $x_n \in Z$ for infinitely many n. In the first case, we can choose a subsequence (x'_n) of (x_n) such that $x'_n \in Y$ for all n in other words we have a sequence in Y. As Y is compact, some subsequence (x''_n) of (x'_n) converges in Y, and thus in $Y \cup Z$. The other case is similar, so in either case some subsequence of (x_n) converges in $Y \cup Z$. This implies that $Y \cup Z$ is compact.

(c) If X has only one point then every sequence converges so X is compact. If X has $n > 1$ points, we can write it in the form $X = Y \cup Z$ where $|Y| = n - 1$ and $|Z| = 1$, so Y and Z are compact by induction, so X is compact by (ii).

(d) Let (w_n) be a sequence in $Y \times Z$, with $w_n = (y_n, z_n)$ say. As Y is compact, some subsequence (y_{n_k}) converges to some $y \in Y$. Put $y'_k = y_{n_k}$ and $z'_k = z_{n_k}$ and $w'_k = (y'_k, z'_k) = w_{n_k}$. As Z is compact, some subsequence z'_{k_j} converges to some point $z \in Z$. Put $y''_j = y'_{k_j}$ and $z''_j = z'_{k_j}$ and $w''_j = (y''_j, z''_j) = w'_{k_j}$. As (y''_j) is a subsequence of the sequence (y'_k) which converges to y, we see that $y''_j \to y$. By assumption we have $z''_j \to z$, so $w''_j \to (y, z)$. Thus, some subsequence of (w_n) converges in $Y \times Z$, proving that $Y \times Z$ is compact as claimed.

(e) As $Z \neq \emptyset$ we can choose a point $a \in Z$. Let $p \colon Y \times Z \to Y$ be defined by $p(y, z) = y$. We have $p(y, a) = y$, which shows that p is surjective. In general, if $f \colon A \to B$ is a surjective continuous map of spaces and A is compact we know that B is compact. As $Y \times Z$ is assumed compact, we deduce that Y is compact.

(f) If $(x, y, z) \in X$ then $x^4 \leq x^4 + y^4 + z^4 = 1$ so $|x| \leq 1$. Similarly, we see that $|y| \leq 1$ and $|z| \leq 1$, which implies that X is bounded. I claim that it is also closed in $\mathbb{R}^3$. Indeed, suppose we have a sequence $a_n = (x_n, y_n, z_n)$ in X converging to some point $a = (x, y, z) \in \mathbb{R}^3$. then $x_n^4 + y_n^4 + z_n^4 = 1$ and $x_n \to x$, $y_n \to y$ and $z_n \to z$, so by the algebra of limits we have

$$x^4 + y^4 + z^4 = \lim(x_n^4 + y_n^4 + z_n^4) = 1,$$

so $a \in X$.

A bounded closed subset of $\mathbb{R}^n$ is compact, so we deduce that X is compact as claimed.

(3)

(a) What does it mean to say that a metric space X is *compact*? **(3 marks)**

(b) Let $f \colon X \to Y$ be a continuous surjective map of metric spaces, where X is compact. Prove that Y is compact. **(6 marks)**

(c) Let Z be a closed subset of a compact space X. Prove that Z is compact. **(6 marks)**

(d) Put $U = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$, and define $g \colon \mathbb{C} \to \mathbb{C}$ by $g(z) = e^z$.

   (i) Is U compact? **(2 marks)**

   (ii) Is $g(U)$ compact? **(4 marks)**

   (iii) Is $g(g(U))$ compact? **(4 marks)**

Justify your answers.

Solution:

(a) A metric space X is compact if for every sequence (x_n) in X there is a subsequence (x_{n_k}) and a point $x \in X$ such that $x_{n_k} \to x$. **[bookwork]**[3]

(b) Let $f\colon X \to Y$ be a continuous surjective map, and suppose that X is compact. Consider a sequence (y_n) in Y. As f is surjective, we can choose $x_n \in X$ for each n such that $f(x_n) = y_n$. As X is compact, there is a subsequence (x_{n_k}) of (x_n) and a point $x \in X$ such that $x_{n_k} \to x$. Put $y = f(x) \in Y$, and note that $y_{n_k} = f(x_{n_k})$. As f is continuous, it follows that $y_{n_k} \to y$. Thus our original sequence has a convergent subsequence, proving that Y is compact. **[bookwork]**[6]

(c) Let X be compact, and let Z be a closed subspace of X. Consider a sequence (z_n) in Z. We can regard this as a sequence in the compact space X, so some subsequence (z_{n_k}) converges to some point $x \in X$. However, Z is closed and z_{n_k} lies in Z for all k and $z_{n_k} \to x$, so x must actually lie in Z. Thus our original sequence has a subsequence that converges to a point in Z, proving that Z is compact. **[bookwork]**[6]

(d) Put $U = \{z \in \mathbb{C} \mid 0 \leq \mathrm{Re}(z) \leq 1\}$, and define $g\colon \mathbb{C} \to \mathbb{C}$ by $g(z) = e^z$. Then U is clearly unbounded and thus not compact; the sequence $i, 2i, 3i, \ldots$ has no convergent subsequence. [2] On the other hand, we can use the fact that $g(x + iy) = e^x(\cos(y) + i\sin(y))$ to see that $g(U) = \{z \in \mathbb{C} \mid 1 \leq |z| \leq e\}$ [2]. This is bounded and closed and thus compact [2]. We can regard g as a continuous surjective map from $g(U)$ to $g(g(U))$ and it follows from (b) that $g(g(U))$ is compact [4] **[unseen]**. **The properties of the complex exponential map are reviewed in lectures and used in several examples.**

(4)

(a) What does it mean to say that a metric space X is *compact*? **(3 marks)**

(b) Let X and Y be compact metric spaces. Prove that $X \times Y$ is compact. **(8 marks)**

(c) Let $f\colon I \to Y$ be a continuous map (where $I = [0, 1]$). Prove that $f(I)$ is closed in Y. **(7 marks)**

(d) Put $X = \mathbb{Z} \times \mathbb{Z}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$, considered as subspaces of the plane $\mathbb{R}^2$.

  (i) Is X compact? **(2 marks)**

  (ii) Is Y compact? **(2 marks)**

  (iii) Is $X \cap Y$ compact? **(3 marks)**

Justify your answers.

Solution:

(a) A metric space X is compact if for every sequence (x_n) in X there is a subsequence (x_{n_k}) and a point $x \in X$ such that $x_{n_k} \to x$. **[bookwork]**[3]

(b) Consider a sequence $z_n = (x_n, y_n)$ in $X \times Y$ [1]. As X is compact, the sequence (x_n) has a convergent subsequence, say $(x_{n_1}, x_{n_2}, \ldots)$ converging to $x \in X$ [1]. We write $x'_m = x_{n_m}$ for convenience, and also put $y'_m = y_{n_m}$ and $z'_m = (x'_m, y'_m) = z_{n_m}$ [1]. Note that $x'_m \to x$ as $m \to \infty$ [1]. Next, observe that Y is compact, so the sequence (y'_m) has a convergent subsequence, say $(y'_{m_1}, y'_{m_2}, \ldots)$ converging to $y \in Y$ [1]. Now put $y''_k = y'_{m_k}$ and $x''_k = x'_{m_k}$ and $z''_k = (x''_k, y''_k)$ [1]. Then $y''_k \to y$ by assumption, and $x''_k \to x$ because (x''_k) is a subsequence of (x'_m), and $x'_m \to x$ [1]. This means that $z''_k = (x''_k, y''_k) \to (x, y)$, so (z''_k) is a convergent subsequence of our original sequence (z_n) [1]. This proves that $X \times Y$ is compact. **[bookwork]**

(c) We know that I is compact [2], and that the image of a compact set under any continuous map is again compact [2]. This means that $f(I)$ is a compact subspace of Y [1]. However, any compact subset of a metric space is automatically closed [2], so $f(I)$ is closed in Y as claimed. **[seen]**

(d) (i) X is unbounded and thus not compact. [2]

  (ii) Y is not closed, and thus is not compact. [2]

  (iii) $X \cap Y$ is a finite set; explicitly,

$$X \cap Y = \{(-1, -1), (-1, 0), (-1, 1), (0, -1), (0, 0), (0, 1), (1, -1), (1, 0), (1, 1)\}.$$

It follows that $X \cap Y$ is compact. [3]

2 Path components

(5)

(a) What does it mean to say that a topological space X is *path-connected*?

(b) Prove that the space S^n is path-connected for all $n > 0$.

(c) Let X be a subset of $\mathbb{R}^n$, and let a be a point in X. What does it mean to say that X is *star-shaped* around a? Show that if X is star-shaped around a, then it is path-connected.

(d) Suppose that $f\colon X \to \mathbb{R}$ is continuous, $f(x)$ is nonzero for all x, and there exist $x_0, x_1 \in X$ with $f(x_0) < 0 < f(x_1)$. Prove that X is not path-connected.

(e) Recall that $GL_3(\mathbb{R})$ is the space of 3×3 invertible matrices over $\mathbb{R}$. Prove that this space is not path-connected.

Solution:

(a) A space X is *path-connected* if for each pair of points $x_0, x_1 \in X$ there exists a continuous map $u\colon I \to X$ such that $u(0) = x_0$ and $u(1) = x_1$.

(b) Suppose that $n > 1$ and that $x_0, x_1 \in S^n$. Suppose first that $x_1 \neq -x_0$, so that the line segment from x_0 to x_1 does not pass through the origin. Thus, if we put $f(t) = (1-t)x_0 + tx_1$ then $f(t) \neq 0$ for all $t \in I$. We can thus define a continuous map $u\colon I \to S^n$ by $u(t) = f(t)/\|f(t)\|$ and this satisfies $u(0) = x_0/\|x_0\| = x_0$ and $u(1) = x_1$ as required.

Now consider the exceptional case where $x_1 = -x_0$. As $n > 0$ the set S^n has more than two points so we can choose a point x_2 that is different from both $-x_0$ and $-x_1$. By the first part of the proof we can define a path u from x_0 to x_2 and a path v from x_1 to x_2 in S^n. This gives a path $w := u * \overline{v}$ from x_0 to x_2.

(c) A subset $X \subseteq \mathbb{R}^n$ is *star-shaped* around a point $a \in X$ if for all $x \in X$, the linear path from x to a (given by the formula $u(t) = (1-t)x + ta$, which is meaningful because x and a are vectors in $\mathbb{R}^n$) lies wholly in X.

Suppose that this holds. For any $x_0, x_1 \in X$ we can let u_0 be the linear path from x_0 to a and let u_1 be the linear path from x_1 to a. Then $u_0 * \overline{u}_1$ is a path from x_0 to x_1, showing that X is path-connected.

(d) Suppose that $f\colon X \to \mathbb{R}$ is continuous, $f(x)$ is nonzero for all x, and there exist $x_0, x_1 \in X$ with $f(x_0) < 0 < f(x_1)$. I claim that there is no continuous path in X from x_0 to x_1, so that X is not path-connected. Indeed, if u is such a path, put $g(t) = f(u(t))$, giving a continuous function $g\colon I \to \mathbb{R}$. We have $g(0) = f(x_0) < 0$ and $g(1) = f(x_1) > 0$. By the Intermediate Value Theorem, there must be some $t \in I$ with $g(t) = 0$, or in other words $f(u(t)) = 0$. However, $u(t) \in X$, and $f(x) \neq 0$ for all $x \in X$ by assumption. This contradiction shows that there can be no such map u.

(e) Consider the map $\det\colon GL_3(\mathbb{R}) \to \mathbb{R}$. As $\det(A)$ is a polynomial expression in the entries of the matrix A, we see that det is continuous. If $A \in GL_3(\mathbb{R})$ then A is invertible, so $\det(A) \neq 0$. The matrices

$$A_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

lie in $GL_3(\mathbb{R})$ and satisfy $\det(A_0) < 0 < \det(A_1)$. It follows from the previous part that $GL_3(\mathbb{R})$ is disconnected.

(6)

(a) Let X be a topological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is an equivalence relation.

(b) Let $f\colon X \to Y$ be a continuous map. Define the induced map $f_*\colon \pi_0(X) \to \pi_0(Y)$, and prove that it is well-defined.

(c) Show that if $f, g\colon X \to Y$ are homotopic maps then $f_* = g_*\colon \pi_0(X) \to \pi_0(Y)$.

(d) Put $X = [-3, -2] \cup [-1, 1] \cup [2, 3]$ and $Y = [0, 1] \cup [2, 10]$, and define $f\colon X \to Y$ by $f(x) = x^2$. Describe the sets $\pi_0(X)$ and $\pi_0(Y)$ and the map $f_*\colon \pi_0(X) \to \pi_0(Y)$.

Solution:

(a) We write $x \sim y$ iff there is a path in X from x to y, in other words a continuous map $s\colon I \to X$ such that $s(0) = x$ and $s(1) = y$. For any $x \in X$ we can define $c_x\colon I \to X$ by $c_x(t) = x$ for all t; this is a path from x to x, proving that $x \sim x$. If $x \sim y$ then there is a path s from x to y and we can define a path $\overline{s}$ from y to x by $\overline{s}(t) = s(1-t)$; this shows that $y \sim x$. If there is also a path r from y to z then we can define a path $s * r$ from x to z by

$$(s * r)(t) = \begin{cases} s(2t) & \text{if } 0 \leq t \leq 1/2 \\ r(2t-1) & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

and this shows that $x \sim z$. Thus $\sim$ is reflexive, symmetric and transitive and thus is an equivalence relation.

(b) Let c be an element of $\pi_0(X)$, in other words a path component in X. For any $x \in c$ we have a point $f(x) \in Y$, and thus a path-component $[f(x)] \in \pi_0(Y)$. If x' is another point in c then $x \sim x'$ so we can choose a path s from x to x' in X. Thus $f \circ s\colon I \to Y$ is a path in Y from $f(x)$ to $f(x')$, so $f(x) \sim f(x')$, so $[f(x)] = [f(x')]$. We can thus define $f_*(c) = [f(x)]$; this is independent of the choice of x and thus is well-defined.

(c) If $f, g\colon X \to Y$ are homotopic then we can chooose a map $h\colon I \to X \to Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all x. If $c \in \pi_0(X)$ we can choose $x \in X$ and note that $f_*(c) = [f(x)]$ and $g_*(c) = [g(x)]$. We can also define a map $s\colon I \to Y$ by $s(t) = h(t, x)$. This gives a path from $s(0) = f(x)$ to $s(1) = g(x)$, so $[f(x)] = [g(x)]$, in other words $f_*(c) = g_*(c)$.

(d) Write

$$\begin{aligned} a &= [-3, -2] \\ b &= [-1, 1] \\ c &= [2, 3] \\ d &= [0, 1] \\ e &= [2, 11] \end{aligned}$$

Then $\pi_0(X) = \{a, b, c\}$ and $\pi_0(Y) = \{c, d\}$. The map $f_*\colon \pi_0(X) \to \pi_0(Y)$ is given by $f_*(a) = f_*(c) = e$ and $f_*(b) = d$.

(7)

(a) Let X be a metric space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is indeed an equivalence relation. **(8 marks)**

(b) Let $f\colon X \to Y$ be a continuous map. Define the function $f_*\colon \pi_0(X) \to \pi_0(Y)$, and check that it is well-defined. **(5 marks)**

(c) Suppose that Y is path-connected and X is not. Show that there do not exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to the identity map id_X. **(6 marks)**

(d) Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$. What can you say about $\det(A)$ when $A \in X$? Show that X is not path-connected. **(6 marks)**

Solution:

(a) Write $x \sim y$ iff there is a path in X from x to y [1], or in other words a continuous map $u\colon I \to X$ such that $u(0) = x$ and $u(1) = y$ [1]. I claim that this is an equivalence relation. Indeed, given $x \in X$ we can define $c_x\colon I \to X$ by $c_x(t) = x$ for all t. This gives a path from x to itself, showing that $\sim$ is reflexive [1]. Next, suppose that $x \sim y$, so there exists a path u from x to y in X. We can then define $\overline{u}(t) = u(1-t)$ to get a path from y to x, showing that $y \sim x$, showing that $\sim$ is symmetric [2]. Finally, suppose we have a path u from x to y, and a path v from y to z. We then define a map $w\colon I \to X$ by

$$w(t) = \begin{cases} u(2t) & \text{if } 0 \leq t \leq 1/2 \\ v(2t-1) & \text{if } 1/2 \leq t \leq 1. \text{[2]} \end{cases}$$

This is well-defined and continuous because $u(1) = y = v(0)$. We have $w(0) = u(0) = x$ and $w(1) = v(1) = z$, so w gives a path from x to z; this proves that $\sim$ is transitive [1]. **[bookwork]**

(b) Let $f\colon X \to Y$ be a continuous map. We define $f_*\colon \pi_0(X) \to \pi_0(Y)$ by $f_*([x]) = [f(x)]$ [1](where $[x]$ is the equivalence class of x under the relation $\sim$). To see that this is well-defined, suppose that $[x_0] = [x_1]$ in $\pi_0(X)$ [1]. This means that $x_0 \sim x_1$, so there is a path $u\colon I \to X$ from x_0 to x_1 [1]. The function $f \circ u\colon I \to Y$ gives a path from $f(x_0)$ to $f(x_1)$ in Y [1], so $[f(x_0)] = [f(x_1)]$ as required [1]. **[bookwork]**

(c) Suppose that Y is path-connected, so $\pi_0(Y)$ has only a single element, which we will call b. Then $f_*\colon \pi_0(X) \to \pi_0(Y)$ must be the constant map with value b, so $g_*f_*\colon \pi_0(X) \to \pi_0(X)$ must be the constant map with value $g_*(b)$. On the other hand, if $gf \simeq 1$ then g_*f_* is the identity. Thus, the identity map of $\pi_0(X)$ is constant, so $\pi_0(X)$ can only have a single element. This means that X is path-connected, contrary to assumption. [6] **[similar examples seen]**

(d) Put $X = \{A \in M_2\mathbb{R} \mid A^2 = A\}$. For $A \in X$ we have $\det(A)^2 = \det(A)$ so $\det(A) \in \{0, 1\}$ [2]. We can thus regard det as a continuous map $X \to \mathbb{R}$ such that $\det(A) \neq 1/2$ for all A. The zero matrix and the identity matrix lie in X, with $\det(0) = 0 < 1/2$ and $\det(I) = 1 > 1/2$. It follows that 0 cannot be connected to I by a path in X, so X is not path-connected. [4] **[similar examples seen]A proposition proved in lectures says that if $f\colon X \to \mathbb{R}$ is nowhere zero and $f(x) < 0$ and $f(y) > 0$ then $x \not\sim y$. A number of examples were discussed, including some where the "missing value" is not zero. In particular, the trace was used to show that $\{A \in M_n\mathbb{R} \mid A^2 = A\}$ is disconnected for $n > 1$.**

3 The fundamental group

These questions involve material that is not covered in the current version of the course.

(8)

(a) Let X be a metric space, and let x_0 and x_1 be points in X. What does it mean to say that two paths from x_0 to x_1 are *pinned homotopic*? Define the set $\pi_1(X; x_0, x_1)$.

(b) Let X be path-connected. Prove that the group $\pi_1(X; x_0)$ is isomorphic to the group $\pi_1(X; x_1)$.

(c) Put $X = \{(w, x, y, z) \in \mathbb{C}^4 \mid w \neq x\ ,\ x \neq y\ ,\ y \neq z\}$, and take $x_0 = (0, 1, 2, 3)$ as the basepoint in X. Calculate $\pi_1(X)$. (You may wish to consider the expression $f(w, x, y, z) = (w, x - w, y - x, z - y)$.)

Solution:

(a) Let X be a metric space, and let x_0 and x_1 be two points in X. Let $u, v\colon I \to X$ be two paths, both of which start at x_0 and end at x_1. We say that u and v are *pinned homotopic* if there exists a map $h\colon I \times I \to X$ such that

- $h(0, t) = u(t)$ for all $t \in I$
- $h(1, t) = v(t)$ for all $t \in I$
- $h(s, 0) = x_0$ for all $s \in I$
- $h(s, 1) = x_1$ for all $s \in I$.

This is an equivalence relation on the set of all paths from x_0 to x_1 in X; the set $\pi_1(X; x_0, x_1)$ is just the set of equivalence classes.

(b) Now suppose that X is path-connected, so we can choose a path u from x_0 to x_1 in X, and put $q = [u] \in \pi_1(X; x_0, x_1)$. If $a \in \pi_1(X; x_0) = \pi_1(X; x_0, x_0)$ then q^{-1} runs from x_1 to x_0, and a runs from x_0 to x_0, and q runs from x_0 to x_1, so $q^{-1}aq$ runs from x_1 to itself. We can thus define a function $f\colon \pi_1(X; x_0) \to \pi_1(X; x_1)$ by $f(a) = q^{-1}aq$. Similarly, we can define $g\colon \pi_1(X; x_1) \to \pi_1(X; x_0)$ by $g(b) = qbq^{-1}$. Clearly $g(f(a)) = qq^{-1}aqq^{-1} = a$ and similarly $f(g(b)) = b$, so f is a bijection with inverse g. Moreover, $f(a)f(a') = q^{-1}aqq^{-1}a'q = q^{-1}aa'q = f(aa')$, so f is a group homomorphism, and thus an isomorphism $\pi_1(X; x_0) \to \pi_1(X; x_1)$.

(c) Define $f\colon X \to \mathbb{C} \times \mathbb{C}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times$ by $f(w, x, y, z) = (w, x - w, y - x, z - y)$ (where $\mathbb{C}^\times$ means $\mathbb{C} \setminus \{0\}$). This is a homeomorphism, with inverse $f^{-1}(a, b, c, d) = (a, a + b, a + b + c, a + b + c + d)$. On the other hand, $\mathbb{C}$ is homotopy equivalent to a point, and $\mathbb{C}^\times$ is homotopy equivalent to S^1; it follows that X is homotopy equivalent to $S^1 \times S^1 \times S^1$, and thus that $\pi_1(X)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

(9)

(a) Let X be a based topological space, and let Y be a subspace of X containing the basepoint. What does it mean to say that Y is a *retract* of X?

(b) Prove that if Y is a retract of X, then $|\pi_1(Y)| \leq |\pi_1(X)|$.

(c) Recall that $\mathbb{R}P^3$ is a subspace of the space $M_4(\mathbb{R})$ of all 4×4 matrices over $\mathbb{R}$, which is homeomorphic to $\mathbb{R}^{16}$. Prove that $\mathbb{R}P^3$ is not a retract of $M_4(\mathbb{R})$.

(d) Recall that $U(3)$ is the space of 3×3 matrices A over $\mathbb{C}$ such that $A^\dagger A = I$. You may assume that for such A we have $\det(A) \in S^1$. Define $j\colon S^1 \to U(3)$ by

$$j(z) = \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

What is $\det(j(z))$? Deduce that $\pi_1(U(3))$ is infinite.

Solution:

(a) Let X be a based topological space, let Y be a subspace of X containing the basepoint, and let $i\colon Y \to X$ be the inclusion map. We say that Y is a *retract* of X if there exists a continuous map $r\colon X \to Y$ such that $r \circ i = \mathrm{id}_Y$, or equivalently $r(y) = y$ for all $y \in Y$.

(b) Suppose that Y is a retract of X. We then have homomorphisms $i_*\colon \pi_1(Y) \to \pi_1(X)$ and $r_*\colon \pi_1(X) \to \pi_1(Y)$ such that $r_* i_* = 1\colon \pi_1(Y) \to \pi_1(Y)$. Now let a and a' be two different elements of $\pi_1(Y)$. Then $r_*(i_*(a)) = a \neq a' = r_*(i_*(a'))$, so clearly $i_*(a)$ cannot be the same as $i_*(a')$. Thus, all the different elements of $\pi_1(Y)$ are mapped to different elements of $\pi_1(X)$, so there must be at least as many elements in $\pi_1(X)$ as there are in $\pi_1(Y)$. In other words, we have $|\pi_1(Y)| \leq |\pi_1(X)|$.

(c) We have $|\pi_1(\mathbb{R}P^3)| = 2$ and $M_4(\mathbb{R}) \simeq \mathbb{R}^{16}$ is contractible so $|\pi_1(M_4(\mathbb{R}))| = 1$. By the previous part, $\mathbb{R}P^3$ cannot be a retract of $M_4(\mathbb{R})$.

(d) It is easy to see that $\det(j(z)) = z$, so $\det \circ j = 1\colon S^1 \to S^1$. It follows that the maps $j_*\colon \pi_1(S^1) \to \pi_1(U(3))$ and $\det_*\colon \pi_1(U(3)) \to \pi_1(S^1)$ satisfy $\det_* \circ j_* = 1\colon \pi_1(S^1) \to \pi_1(S^1)$. By the logic of part (b) we see that $|\pi_1(U(3))| \geq |\pi_1(S^1)| = |\mathbb{Z}| = \infty$.

4 Homotopy equivalence

(10)

(a) Let $f, g\colon X \to Y$ be continuous maps between topological spaces. What does it mean to say that f is *homotopic* to g?

(b) Let X and Y be topological spaces. What does it mean to say that X and Y are *homotopy equivalent*?

(c) Show that if X and Y are homotopy equivalent then there is a bijection between the sets of path-components $\pi_0(X)$ and $\pi_0(Y)$.

(d) Consider the cross $X = \{(x, 0) \mid -1 \leq x \leq 1\} \cup \{(0, y) \mid -1 \leq y \leq 1\}$, and let $C = \mathbb{R}^2 \setminus X$ be its complement. Prove that C is homotopy equivalent to S^1.

Solution:

(a) We say that f and g are homotopic if there exists a continuous map $h\colon I \times X \to Y$ such that $h(0, x) = f(x)$ and $h(1, x) = g(x)$ for all $x \in X$.

(b) We say that spaces X and Y are homotopy equivalent if there are maps $f\colon X \to Y$ and $g\colon Y \to X$ such that gf is homotopic to 1_X and fg is homotopic to 1_Y.

(c) For any map $u\colon X \to Y$ we have an induced function $u_*\colon \pi_0(X) \to \pi_0(Y)$, given by $u_*\langle x\rangle = \langle u(x)\rangle$ for all $x \in X$. These maps satisfy $1_* = 1$ and $(vu)_* = v_*u_*$, and $u'_* = u_*$ if u' is homotopic to u. If f, g are as above we then have maps $f_*\colon \pi_0(X) \to \pi_0(Y)$ and $g_*\colon \pi_0(Y) \to \pi_0(X)$, satisfying

$$f_*g_* = (fg)_* = (1_Y)_* = 1_{\pi_0(Y)}$$

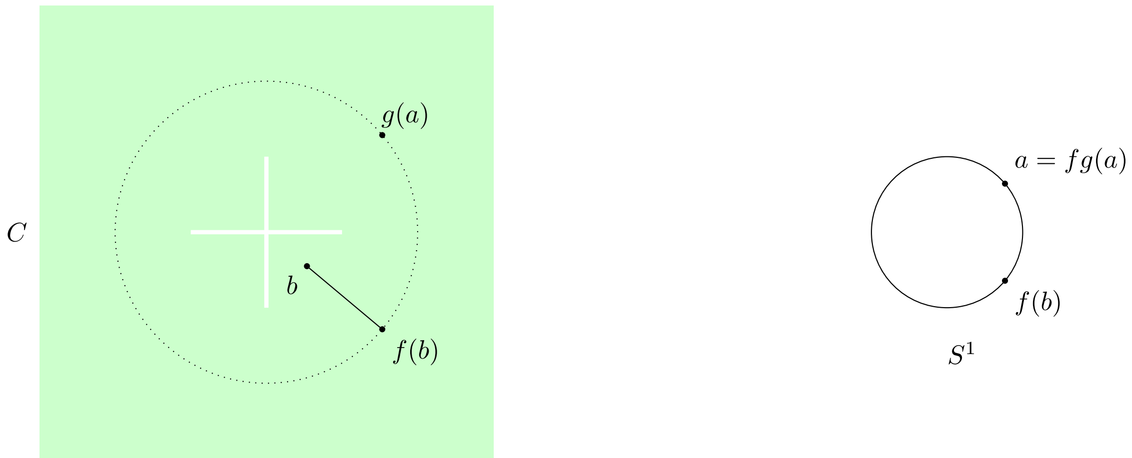
$$g_*f_* = (gf)_* = (1_X)_* = 1_{\pi_0(X)}.$$

Thus g_* is an inverse for f_*, so f_* is a bijection.

(d) Define maps as follows:

$$\begin{array}{ll} f\colon C \to S^1 & f(x,y) = (x,y)/\sqrt{x^2+y^2} \\ g\colon S^1 \to C & g(x,y) = (2x, 2y) \\ h\colon I \times C \to C & h(t,x,y) = (1-t)(x,y) + t(x,y)/\sqrt{x^2+y^2}. \end{array}$$

Then $fg = 1_{S^1}$, and h is a (linear) homotopy from 1_C to gf, so f is a homotopy equivalence.



(11) Consider a metric space X.

(a) (i) What does it mean to say that a subset U of X is open?
(ii) What does it mean to say that a subset F of X is closed?

(b) Show that a subset $F \subseteq X$ is closed iff for every sequence (x_n) in F that converges to a point $x \in X$, we actually have $x \in F$.

(c) Explain what it means for a subset $A \subseteq X$ to be compact. Show that if A is compact and $f\colon X \to Y$ is continuous then $f(A)$ is compact.

(d) Prove that the space $[0, 1]$ is compact. Show that there is a continuous bijection $g\colon [-1, -1/2) \cup [1/2, 1] \to [0, 1]$; can it be chosen to be a homeomorphism?

Solution:

(a) We say that $U \subseteq X$ is *open* if for each point $x \in U$, there exists $\epsilon > 0$ such that the open ball $OB(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}$ is contained in U.

We say that $F \subseteq X$ is closed if the complement $X \setminus F$ is open.

(b) Suppose that F is closed, and that (x_n) is a sequence in F converging to a point $x \in X$. I claim that $x \in F$. If not, then x lies in the open set $X\setminus F$, so there exists $\epsilon > 0$ such that $\overset{\circ}{B}_\epsilon(x) \subseteq X\setminus F$, or equivalently $\overset{\circ}{B}_\epsilon(x) \cap F = \emptyset$. Because $x_n \to x$, there exists N such that $d(x_n, x) < \epsilon$ when $n \geq N$, or in other words $x_n \in \overset{\circ}{B}_\epsilon(x)$ when $n \geq N$. On the other hand, we have $x_n \in F$ for all n by assumption, so for $n \geq N$ we have $x_n \in \overset{\circ}{B}_\epsilon(x) \cap F = \emptyset$, which is impossible. Thus $x \in F$ after all.

Conversely, suppose that F satisfies the condition on sequences; we need to prove that F is closed, or equivalently that $X \setminus F$ is open. If not, then there exists $x \in X \setminus F$ such that $\mathring{B}_\epsilon(x)$ is not contained in $X \setminus F$ for any $\epsilon > 0$. In particular, $\mathring{B}_{1/n}(x)$ is not contained in $X \setminus F$, so we can choose a point $x_n \in \mathring{B}_{1/n}(x) \cap F$. As $x_n \in \mathring{B}_{1/n}(x)$ we have $d(x_n, x) < 1/n$ so $x_n \to x$. Thus (x_n) is a sequence in F converging to the point x outside F, contradicting the condition on sequences.

(c) Not written

(d) Not written

(12)

(a) What does it mean to say that a topological space X is *homotopy equivalent* to a metric space Y? Show that the relation of homotopy equivalence is an equivalence relation.

(b) What does it mean for a space to be (a) *contractible* and (b) *path connected*? Show that any contractible space is path connected. Is the reverse implication true?

(c) Consider the rational comb space

$$X = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0 \text{ or } x \in \mathbb{Q}\}.$$

Show that X is homotopy equivalent to the upper half plane $Y = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, and deduce that X is contractible.

Solution:

(a) We say that X is homotopy equivalent to Y if there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$ (where $p \simeq q$ means that p is homotopic to q).

Clearly any space X is homotopy equivalent to itself, because we can take $f = g = \mathrm{id}_X$.

If X is homotopy equivalent to Y then by reversing the rôles of f and g we see that Y is homotopy equivalent to X.

Now suppose that X is homotopy equivalent to Y and that Y is homotopy equivalent to Z. We can then choose maps f and g as above, and also maps $u\colon Y \to Z$ and $v\colon Z \to Y$ such that $uv \simeq \mathrm{id}_Z$ and $vu \simeq \mathrm{id}_Y$. These give maps $uf\colon X \to Z$ and $gv\colon Z \to X$ such that

$$(uf)(gv) = u(fg)v \simeq u \,\mathrm{id}_Y\, v = uv \simeq \mathrm{id}_Z$$

$$(gv)(uf) = g(vu)f \simeq g \,\mathrm{id}_Y\, f = gf \simeq \mathrm{id}_X,$$

so X is homotopy equivalent to Z.

This shows that the relation of homotopy equivalence is an equivalence relation.

(b) A space X is contractible if it is equivalent to the one-point space $\{0\}$. It is path connected if for any two points $x, y \in X$ there is a path $s\colon I \to X$ with $s(0) = x$ and $s(1) = y$.

Suppose that X is contractible, so we have maps $f\colon X \to \{0\}$ and $g\colon \{0\} \to X$ and a homotopy $h\colon \mathrm{id}_X \simeq gf$. Write $a = g(0) \in X$. Note that we must have $f(x) = 0$ for all $x \in X$, because there are no other points in $\{0\}$ that $f(x)$ could be. Thus $gf(x) = a$ for all x. As h is a homotopy from 1 to gf, we have $h(0, x) = x$ and $h(1, x) = a$ for all x. Thus, for any point $x \in X$ we can define a path $s_x\colon I \to X$ by $s_x(t) = h(t, x)$. This starts at x and ands at a. If y is any other point in X we can take the join of s_x with the reverse of s_y to get a path from x to y. Thus X is path connected.

On the other hand, a path connected space need not be contractible. For example, the space S^1 is path-connected (we can define a path from $e^{i\theta}$ to $e^{i\phi}$ by $s(t) = e^{i((1-t)\theta + t\phi)}$) but not contractible (because $H_1(S^1) \neq 0$).

(c) Let $i\colon Y \to X$ be the inclusion, and define $r\colon X \to Y$ by $r(x, y) = (x, \max(0, y))$. We then have $rj = 1$. I claim that if $(x, y) \in X$ then the line segment joining (x, y) to $jr(x, y)$ is contained in X. If $y \geq 0$ then $rj(x, y) = (x, y)$ and the claim is clear. If $y < 0$ then (as $(x, y) \in X$) we must have $x \in \mathbb{Q}$. We also have $rj(x, y) = (x, 0)$ so the line segment in question is the set of points (x, w) with $y \leq w \leq 0$. As $x \in \mathbb{Q}$, all these points lie in X as required. Thus rj is linearly homotopic to id_X, which implies that j is a homotopy equivalence.

The set Y is convex and thus contractible, in other words homotopy equivalent to a point. As homotopy equivalence is an equivalence relation, we deduce that X is also homotopy equivalent to a point.

(13)

(a) What does it mean to say that a metric space X is *homotopy equivalent* to a metric space Y? Show that the relation of homotopy equivalence is an equivalence relation.

(b) What does it mean for a space to be (i) *contractible* and (ii) *path connected*? Show that any contractible space is path connected. Is the reverse implication true?

(c) Consider the rational comb space

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Show that X is homotopy equivalent to the upper half plane $Y = \{(x, y) \in \mathbb{R}^2 \mid y \geq 0\}$, and deduce that X is contractible.

Solution:

(a) We say that X is homotopy equivalent to Y if there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$ (where $p \simeq q$ means that p is homotopic to q).

Clearly any space X is homotopy equivalent to itself, because we can take $f = g = \mathrm{id}_X$.

If X is homotopy equivalent to Y then by reversing the rôles of f and g we see that Y is homotopy equivalent to X.

Now suppose that X is homotopy equivalent to Y and that Y is homotopy equivalent to Z. We can then choose maps f and g as above, and also maps $u\colon Y \to Z$ and $v\colon Z \to Y$ such that $uv \simeq \mathrm{id}_Z$ and $vu \simeq \mathrm{id}_Y$. These give maps $uf\colon X \to Z$ and $gv\colon Z \to X$ such that

$$(uf)(gv) = u(fg)v \simeq u\,\mathrm{id}_Y\,v = uv \simeq \mathrm{id}_Z$$

$$(gv)(uf) = g(vu)f \simeq g\,\mathrm{id}_Y\,f = gf \simeq \mathrm{id}_X,$$

so X is homotopy equivalent to Z.

This shows that the relation of homotopy equivalence is an equivalence relation.

(b) A space X is contractible if it is equivalent to the one-point space $\{0\}$. It is path connected if for any two points $x, y \in X$ there is a path $s\colon I \to X$ with $s(0) = x$ and $s(1) = y$.

Suppose that X is contractible, so we have maps $f\colon X \to \{0\}$ and $g\colon \{0\} \to X$ and a homotopy $h\colon \mathrm{id}_X \simeq gf$. Write $a = g(0) \in X$. Note that we must have $f(x) = 0$ for all $x \in X$, because there are no other points in $\{0\}$ that $f(x)$ could be. Thus $gf(x) = a$ for all x. As h is a homotopy from 1 to gf, we have $h(0, x) = x$ and $h(1, x) = a$ for all x. Thus, for any point $x \in X$ we can define a path $s_x\colon I \to X$ by $s_x(t) = h(t, x)$. This starts at x and ands at a. If y is any other point in X we can take the join of s_x with the reverse of s_y to get a path from x to y. Thus X is path connected.

On the other hand, a path connected space need not be contractible. For example, the space S^1 is path-connected (we can define a path from $e^{i\theta}$ to $e^{i\phi}$ by $s(t) = e^{i((1-t)\theta + t\phi)}$) but not contractible (because $\pi_1(S^1) \neq \{e\}$).

(c) Let $i\colon Y \to X$ be the inclusion, and define $r\colon X \to Y$ by $r(x, y) = (x, \max(0, y))$. We then have $rj = 1$. I claim that if $(x, y) \in X$ then the line segment joining (x, y) to $jr(x, y)$ is contained in X. If $y \geq 0$ then $rj(x, y) = (x, y)$ and the claim is clear. If $y < 0$ then (as $(x, y) \in X$) we must have $x \in \mathbb{Q}$. We also have $rj(x, y) = (x, 0)$ so the line segment in question is the set of points (x, w) with $y \leq w \leq 0$. As $x \in \mathbb{Q}$, all these points lie in X as required. Thus rj is linearly homotopic to id_X, which implies that j is a homotopy equivalence.

The set Y is convex and thus contractible, in other words homotopy equivalent to a point. As homotopy equivalence is an equivalence relation, we deduce that X is also homotopy equivalent to a point.

(14)

(a) Let X be a subspace of $\mathbb{R}^n$, and let a be a point in X.

(i) Explain what it means for X to be *star-shaped* around a. **(4 marks)**

(ii) Prove that if X is star-shaped around a, then X is contractible. **(4 marks)**

(b) (i) Suppose that $\alpha, \beta > 0$ and that $0 \leq t \leq 1$. Show that $\alpha t + \beta(1 - t)$ is strictly greater than zero. **(3 marks)**

(ii) Suppose that $\gamma, \delta, \epsilon > 0$ and that $0 \leq t \leq 1$. Show that $\gamma t^2 + \delta t(1-t) + \epsilon(1-t)^2$ is strictly greater than zero. **(3 marks)**

(iii) Consider a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2\mathbb{R}$. Put $\lambda = \operatorname{trace}(A)$ and $\mu = \det(A)$. Express $\operatorname{trace}((1-t)I + tA)$ and $\det((1-t)I + tA)$ in terms of λ, μ and t. **(6 marks)**

(iv) Put $X = \{A \in M_2\mathbb{R} \mid \det(A) > 0 \text{ and } \operatorname{trace}(A) > 0\}$. Prove that X is contractible. **(5 marks)**

Solution:

(a) (i) We say that X is star-shaped around a if for each $t \in I$ and $x \in X$, the point $(1-t)x + ta$ lies in X. Equivalently, X is star-shaped around a if every linear path starting in X and ending at a lies wholly in X. [4]

(ii) Suppose that X is star-shaped around a. We can then define a map $h\colon I \times X \to X$ by $h(t,x) = (1-t)x + ta$. We have $h(0,x) = x$ and $h(1,x) = a$ for all $x \in X$, so this gives a contraction of X. [4]

(b) (i) Suppose that $\alpha, \beta > 0$ and $0 \leq t \leq 1$. Then αt and $\beta(1-t)$ are both greater than or equal to 0. Moreover, αt is only zero when $t = 0$, and $\beta(1-t)$ is only zero when $t = 1$. Thus, for any t, at least one of the two terms is strictly positive, and thus $\alpha t + \beta(1-t) > 0$. [3]

(ii) Suppose that $\gamma, \delta, \epsilon > 0$ and that $0 \leq t \leq 1$. Then γt^2, $\delta t(1-t)$ and $\epsilon(1-t)^2$ are all greater than or equal to zero. The first one is strictly greater than zero unless $t = 0$, and the last one is strictly greater than zero unless $t = 1$. Thus, for all t, at least one term is strictly positive, so their sum is strictly positive. [3] **[unseen]**

(iii) We have $\lambda = a + d$ and $\mu = ad - bc$ and

$$(1-t)I + tA = \begin{bmatrix} 1-t+ta & tb \\ tc & 1-t+td \end{bmatrix},$$

so

$$\begin{aligned} \operatorname{trace}((1-t)I + tA)) &= (ta + 1 - t) + (td + 1 - t) \\ &= (a+d)t + 2(1-t) \\ &= \lambda t + 2(1-t)[3] \end{aligned}$$

and

$$\begin{aligned} \det((1-t)I + tA)) &= (ta + 1 - t)(td + 1 - t) - t^2 bc \\ &= (ad - bc)t^2 + (a+d)t(1-t) + (1-t)^2 \\ &= \mu t^2 + \lambda t(1-t) + (1-t)^2.[3] \end{aligned}$$

(iv) Suppose that $A \in X$, so $\lambda, \mu > 0$, and suppose that $t \in I$. As $\lambda > 0$ and $2 > 0$, part (a) tells us that $\lambda t + 2(1-t) > 0$, so $\operatorname{trace}((1-t)I + tA) > 0$. As $\mu > 0$, $\lambda > 0$ and $1 > 0$, part (b) tells us that $\mu t^2 + \lambda t(1-t) + (1-t)^2 > 0$, so $\det((1-t)I + tA) > 0$. This shows that $(1-t)I + tA \in X$, so X is star-shaped around I, and thus contractible. [5]

(15)

(a) Given metric spaces X, Y and continuous maps $f, g\colon X \to Y$, what does it mean for f and g to be homotopic? **(3 marks)**

(b) Show that if Y is contractible, then any two maps $f, g\colon X \to Y$ are homotopic. **(7 marks)**

(c) Show that if X is contractible and Y is path-connected, then any two maps $f, g\colon X \to Y$ are homotopic. **(10 marks)**

(d) Regard S^1 as $\{z \in \mathbb{C} \mid |z| = 1\}$, and put $T = S^1 \times S^1$. Define $f\colon T \to T$ by $f(z,w) = (iz, -iw)$. Prove that f is homotopic to the identity map. **(5 marks)**

Solution:

(a) Maps $f, g\colon X \to Y$ are homotopic iff there is a continuous map $h\colon I \times X \to Y$ such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$ for all $x \in X$. [3] **[bookwork]**

(b) Suppose that Y is contractible, so we have a point $b \in Y$ and a map $m\colon I \times Y \to Y$ with $m(0,y) = y$ and $m(1,y) = b$ for all $y \in Y$ [2]. Let $c_b\colon X \to Y$ be the constant map with value b. Define $h\colon I \times X \to Y$ by $h(t,x) = m(t, f(x))$ [2]. This has $h(0,x) = f(x)$ and $h(1,x) = m(1, f(x)) = b = c_b(x)$, showing that f is homotopic to c_b [1]. Similarly, g is homotopic to c_b and thus to f [2]. **[seen]**

(c) Now suppose instead that X is contractible, so we have a point $a \in X$ and a map $n\colon I \times X \to X$ with $n(0,x) = x$ and $n(1,x) = a$ for all $x \in X$ [2]. Given $f\colon X \to Y$ we put $k(t,x) = f(n(t,x))$ [2]. This has $k(0,x) = f(x)$ and $k(1,x) = f(a) = c_{f(a)}(x)$, showing that $f \simeq c_{f(a)}$ [1]. Similarly $g \simeq c_{g(a)}$ [1]. Finally, as Y is path-connected, we can choose a path $u\colon I \to Y$ with $u(0) = f(a)$ and $u(1) = g(a)$ [2]. Put $l(t,x) = u(t)$; then $l(0,x) = u(0) = f(a) = c_{f(a)}(x)$ and $l(1,x) = u(1) = g(a) = c_{g(a)}(x)$, so $f \simeq c_{f(a)} \simeq c_{g(a)} \simeq g$ as required. [2] **The case $X = I$ is done in lectures, but otherwise this is unseen.**

(d) Define $h(t,z,w) = (e^{i\pi t/2}z, e^{-i\pi t/2}w)$. Then $h(0,z,w) = (z,w)$ and

$$h(1,z,w) = (e^{i\pi/2}z, e^{-i\pi/2}w) = (iz, -iw) = f(z,w),$$

showing that f is homotopic to the identity. [5] **[similar examples seen]**

(16) Let E be the figure eight space, so $E = E_- \cup E_+$ where $E_\pm$ is the circle of radius one centred at $(\pm 1, 0)$.

(a) Prove that E is not homotopy equivalent to the torus. **(4 marks)**

(b) Put $A = \{(1,0), (-1,0)\}$ and $X = \mathbb{R}^2 \setminus A$. Sketch a proof that X is homotopy equivalent to E. **(5 marks)**

(c) Put $B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 = 1\ ,\ y = 0\}$ and $Y = \mathbb{R}^3 \setminus B$. Deduce that Y is homotopy equivalent to E. **(4 marks)**

(d) Put $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 = 1\ ,\ y = xz\}$ and $Z = \mathbb{R}^3 \setminus C$. Deduce that Z is homotopy equivalent to E. You may wish to consider the expression

$$(x, \mathrm{rot}_{\pi x/4}(y,z)) = (x, \cos(\pi x/4)y - \sin(\pi x/4)z, \sin(\pi x/4)y + \cos(\pi x/4)z).$$

(12 marks)

Solution:

(a) The torus T has $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$ [1], which is abelian [1], but $\pi_1(E)$ is nonabelian [1], so $\pi_1(T) \not\simeq \pi_1(E)$, so T is not homotopy equivalent to E [1]. **[similar examples seen]**

(b) Let $g\colon X \to E$ be as illustrated in the following diagram:

C2def.eps[2]

Let $f\colon E \to X$ be the inclusion. Then $gf = \mathrm{id}_E$ [1], and the line joining $fg(a)$ to a lies wholly in X so fg is linearly homotopic to id_X [2]. This shows that X is homotopy equivalent to E. **[bookwork]**

(c) We observe that $B = A \times \mathbb{R}$ [2] and so $Y = X \times \mathbb{R}$ [1], and $\mathbb{R}$ is contractible so $Y \simeq X$ [1].

More explicitly, define $p\colon X \to Y$ by $p(x,y) = (x,y,0)$ and $q\colon Y \to X$ by $q(x,y,z) = (x,y)$. Then $qp = \mathrm{id}_X$ and pq is linearly homotopic to id_Y, so $Y \simeq X \simeq E$.

(d) We will show that Z is homeomorphic to Y [2], and so homotopy equivalent to Y, X and E [1]. We define $r\colon \mathbb{R}^3 \to \mathbb{R}^3$ by $r(x,y,z) = (x, \mathrm{rot}_{\pi x/4}(y,z))$ [1]. This is a homeomorphism, with inverse $s(x,y,z) = (x, \mathrm{rot}_{-\pi x/4}(y,z))$ [2]. The points in C have the form $(1,y,y)$ or $(-1,y,-y)$ [1]. We have $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, so $r(1,y,y) = (1,0,\sqrt{2}y) \in B$ [1]. Similarly, we have $\cos(-\pi/4) = 1/\sqrt{2}$ and $\sin(-\pi/4) = -1/\sqrt{2}$, so $r(-1,y,-y) = (-1,0,-\sqrt{2}y) \in B$ [1]. Using this, we see that $r(C) = B$ [1]and so r induces a homeomorphism

$$Z = \mathbb{R}^3 \setminus C \simeq \mathbb{R}^3 \setminus r(C) = \mathbb{R}^3 \setminus B = Y$$

as required [2]. **[unseen]**

5 Abelian groups and chain complexes

(17)

(a) In the context of Abelian groups, define the terms

- homomorphism **(2 marks)**
- subgroup **(2 marks)**
- kernel **(2 marks)**
- image. **(2 marks)**

(b) Let A and B be Abelian groups, and let $\phi\colon A \to B$ be a homomorphism. Prove that

(i) The kernel of ϕ is a subgroup of A **(3 marks)**

(ii) The kernel of ϕ is a subgroup of the kernel of the homomorphism 2ϕ. **(2 marks)**

(c) Let $\phi\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}/12$ be the homomorphism defined by

$$\phi(n, m) = (3n, (2n + 4m \bmod 12)).$$

Give an isomorphism $\psi\colon \mathbb{Z} \to \ker(\phi)$. **(6 marks)**

(d) Let A be a finite Abelian group, and let B be a free Abelian group. Prove that if $\phi\colon A \to B$ is a homomorphism, then $\phi = 0$. **(6 marks)**

Solution:

(a) (i) A homomorphism from an Abelian group A to an Abelian group B is a function $\phi\colon A \to B$ such that $\phi(a+a') = \phi(a)+\phi(a')$ for all $a, a' \in A$ [2] (from which it follows that $\phi(0) = 0$ and $\phi(a-a') = \phi(a)-\phi(a')$).

(ii) A subgroup of A is a subset $C \subseteq A$ with the property that $0 \in C$, and $-c \in C$ whenever $c \in C$, and $c + c' \in C$ whenever $c, c' \in C$. [2]

(iii) The kernel of a homomorphism $\phi\colon A \to B$ is $\{a \in A \mid \phi(a) = 0\}$. [2]

(iv) The image of a homomorphism $\phi\colon A \to B$ is $\{\phi(a) \mid a \in A\} = \{b \in B \mid b = \phi(a) \text{ for some } a \in A\}$. [2]

[**bookwork**]

(b) (i) First, we have $\phi(0_A) = 0_B$ so $0_A \in \ker(\phi)$ [1]. Next, suppose we have $c \in \ker(\phi)$, so $\phi(c) = 0$. We then have $\phi(-c) = -\phi(c) = -0 = 0$, so $-c \in \ker(\phi)$ [1]. Finally, suppose we have another element $c' \in \ker(\phi)$, so that $\phi(c') = 0$. Then $\phi(c+c') = \phi(c) + \phi(c') = 0 + 0 = 0$, so $c + c' \in \ker(\phi)$ [1]. This proves that $\ker(\phi)$ is a subgroup. [**bookwork**]

(ii) By the first part we know that $\ker(\phi)$ and $\ker(2\phi)$ are subgroups; we need only check that $\ker(\phi) \subseteq \ker(2\phi)$ [1]. If $a \in \ker(\phi)$ then $\phi(a) = 0$ so $(2\phi)(a) = 2\phi(a) = 2.0 = 0$, so $a \in \ker(2\phi)$ as required [1].

(c) We have $\phi(n, m) = 0$ iff $3n = 0$ and $2n + 4m = 0 \pmod{12}$ [2]. This is clearly equivalent to $n = 0$ and $4m = 0 \pmod{12}$, which means that $n = 0$ and m is divisible by 3 [2]. We can thus define an isomorphism $f\colon \mathbb{Z} \to \ker(\phi)$ by $f(k) = (0, 3k)$ [2]. [**similar examples seen**]

(d) Let A be a finite Abelian group, and let B be a free Abelian group, say $B = \mathbb{Z}[D]$ for some set D. Suppose that $a \in A$ and $\phi(a) = n_1[d_1] + \ldots + n_r[d_r]$ say, for some integers $n_1, \ldots, n_r$ and distinct elements $d_1, \ldots, d_r \in D$ [2]. As A is finite we know that $ma = 0$ for some $m > 0$ [1]. We thus have

$$mn_1[d_1] + \ldots + mn_r[d_r] = m\phi(a) = \phi(ma) = \phi(0) = 0.[1]$$

As the d_i are distinct, the only way this can happen is if $n_1 = \ldots = n_r = 0$, so $\phi(a) = 0$ [1]. This holds for all $a \in A$, so ϕ must be the zero homomorphism as claimed. [1]

(18) **2018-19 Q4:** Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps.

(a) Define what is meant by saying that the above sequence is short exact. **(3 marks)**

Now recall that a *snake* for the above sequence is a system (c, w, v, u, a) such that

- $c \in H_n(W)$;
- $w \in Z_n(W)$ is a cycle such that $c = [w]$;
- $v \in V_n$ is an element with $p(v) = w$;
- $u \in Z_{n-1}(U)$ is a cycle with $i(u) = d(v) \in V_{n-1}$;
- $a = [u] \in H_{n-1}(U)$.

(b) Prove that for each $c \in H_n(W)$ there is a snake starting with c. **(8 marks)**

(c) Prove that if two snakes have the same starting point, then they also have the same endpoint. **(10 marks)**

(d) Suppose that the differential $d\colon V_{n+1} \to V_n$ is surjective. Show that any snake starting in $H_n(W)$ ends with zero. **(4 marks)**

Solution:

(a) The map i is injective, the map p is surjective, and the image of i is the same as the kernel of p. [3] **[Bookwork]**

(b) Consider an element $c \in H_n(W)$. As $H_n(W) = Z_n(W)/B_n(W)$ by definition, we can certainly choose $w \in Z_n(W)$ such that $c = [w]$ [1]. As the sequence $U \xrightarrow{i} V \xrightarrow{p} W$ is short exact, we know that $p\colon V_n \to W_n$ is surjective, so we can choose $v \in V_n$ with $p(v) = w$ [1]. As p is a chain map we have $p(d(v)) = d(p(v)) = d(w) = 0$ (the last equation because $w \in Z_n(W)$) [1]. This means that $d(v) \in \ker(p)$, but $\ker(p) = \mathrm{img}(i)$ because the sequence is exact, so we have $u \in U_{n-1}$ with $i(u) = d(v)$ [2]. Note also that $i(d(u)) = d(i(u)) = d(d(v)) = 0$ (because i is a chain map and $d^2 = 0$) [1]. On the other hand, exactness means that i is injective, so the relation $i(d(u)) = 0$ implies that $d(u) = 0$ [1]. This shows that $u \in Z_{n-1}(U)$, so we can put $a = [u] \in H_{n-1}(U)$ [1]. We now have a snake (c, w, v, u, a) starting with c as required. **[Bookwork]**

(c) Suppose we have two snakes that start with c. We can then subtract them to get a snake $(0, w, v, u, a)$ starting with 0 [1]. It will be enough to show that this ends with 0 as well, or equivalently that $a = 0$ [1]. The first snake condition says that $[w] = 0$, which means that $w = d(w')$ for some $w' \in W_{n+1}$ [1]. Because p is surjective we can also choose $v' \in V_{n+1}$ with $w' = p(v')$ [1], and this gives $w = d(w') = d(p(v')) = p(d(v'))$ [1]. The next snake condition says that $p(v) = w$. We can combine these facts to see that $p(v - d(v')) = 0$, so $v - d(v') \in \ker(p) = \mathrm{img}(i)$[1]. We can therefore find $u' \in U_n$ with $v - d(v') = i(u')$ [1]. We can apply d to this using $d^2 = 0$ and $di = id$ to get $d(v) = i(d(u'))$ [1]. On the other hand, the third snake condition tells us that $d(v) = i(u)$. Subtracting these gives $i(u - d(u')) = 0$, but i is injective, so $u = d(u')$, so $u \in B_{n-1}(U)$ [1]. The final snake condition now says that $a = [u] = u + B_{n-1}(U)$, but $u \in B_{n-1}(U)$ so $a = [u] = 0$ [1]. **[Bookwork]**

(d) Now suppose that $d\colon V_{n+1} \to V_n$ is surjective. As $d^2 = 0$ this means that $d\colon V_n \to V_{n-1}$ is zero. Now suppose we have a snake (c, w, v, u, a) with $c \in H_n(W)$ so $v \in V_n$. The condition $i(u) = d(v)$ now gives $i(u) = 0$, but i is injective so $u = 0$, so $a = [u] = 0$. [4] **[Unseen]**

6 Singular chains

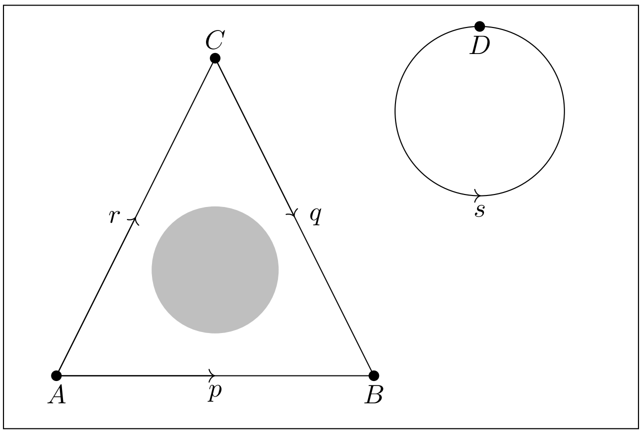
(19) Let X be a topological space.

(a) Let $c\colon \Delta_1 \to X$ be a constant path. Prove that c is homologous to 0.

(b) Let $s\colon \Delta_1 \to X$ be a path. Define the reversed path $\overline{s}$, and prove that $\overline{s}$ is homologous to $-s$.

(c) Let $r, s\colon \Delta_1 \to X$ be paths such that $r(e_1) = s(e_0)$. Write down a path $u\colon \Delta_1 \to X$ and prove that u is homologous to $r + s$.

(d) Let X be the complement of the shaded disc in the diagram below. Write down a path $u\colon \Delta_1 \to X$ such that u is homologous to $2p - 2q - 2r + s$.



Solution:

(a) As c is constant, there is a point $x \in X$ with $c(t) = x$ for all $t \in \Delta_1$. Define $d\colon \Delta_2 \to X$ by $d(t) = x$ for all $t \in \Delta_2$. Then $d\delta_0 = d\delta_1 = d\delta_2 = c\colon \Delta_1 \to X$, so $\partial(d) = c - c + c = c$, so $[c] = [0]$ in $H_1(X)$.

(b) First, we put $\overline{s}(t_0, t_1) = s(t_1, t_0)$. Next, we put $a = s(1, 0)$, and we define $r\colon \Delta_2 \to X$ by $r(t_0, t_1, t_2) = s(t_0+t_2, t_1)$. Then $r\delta_0(t_0, t_1) = r(0, t_0, t_1) = s(t_1, t_0) = \overline{s}(t_0, t_1)$ and $r\delta_1(t_0, t_1) = r(t_0, 0, t_1) = s(t_0 + t_1, 0) = s(1, 0) = a$ and $r\delta_2(t_0, t_1) = r(t_0, t_1, 0) = s(t_0, t_1)$, so $\partial(r) = \overline{s} - c_a + s$ so $\overline{s} + s - c_a \in B_1(X)$. From (i) we also know that $c_a \in B_1(X)$ so $\overline{s} + s \in B_1(X)$.

(c) We define a path $u = r * s\colon \Delta_1 \to X$ by

$$u(t_0, t_1) = \begin{cases} r(t_0 - t_1, 2t_1) & \text{if } t_0 \geq t_1 \\ s(2t_0, t_1 - t_0) & \text{if } t_0 \leq t_1. \end{cases}$$

This is well-defined and continuous (by closed patching) because $r(e_1) = s(e_0)$. We also define $w\colon \Delta_2 \to X$ by

$$w(t_0, t_1, t_2) = \begin{cases} r(t_0 - t_2, t_1 + 2t_2) & \text{if } t_0 \geq t_2 \\ s(2t_0 + t_1, t_2 - t_0) & \text{if } t_0 \leq t_2. \end{cases}$$

When $t_0 = t_2$ the first clause gives $r(e_1)$ and the second gives $s(e_0)$, and these are the same by assumption, so w is well-defined and continuous (by closed patching again). Now $w\delta_0(t_0, t_1) = w(0, t_0, t_1) = s(t_0, t_1)$ and $w\delta_1(t) = w(t_0, 0, t_1) = u(t_0, t_1)$ and $w\delta_2(t) = w(t_0, t_1, 0) = r(t_0, t_1)$, so $\partial(w) = s - u + r$, so $u + B_1(X) = (r + B_1(X)) + (s + B_1(X))$ as required.

(d) The path s in the diagram is linearly homotopic in X to the constant map with value D. As $s(e_0) = s(e_1) = D$ we see that this is a pinned homotopy, so s is homologous to a constant map and thus to 0. As p ends where $\overline{q}$ starts, and $\overline{q}$ ends where $\overline{r}$ starts, we can join these together to get a path $v = (p * \overline{q}) * \overline{r}$. This has $v = p - q - r$ (mod $B_1(X)$). It starts and ends at the same place, so we can form $u = v * v$, and this has $u = 2v = 2p - 2q - 2r + s$ (mod $B_1(X)$) as required.

(20)

(a) Let X be a topological space.

(i) Define the groups $C_n(X)$ for all nonnegative integers n. **(2 marks)**

(ii) Define the homomorphisms ∂_n. **(3 marks)**

(iii) Prove that $\partial_1 \circ \partial_2 = 0$. **(3 marks)**

(iv) Define the groups $H_n(X)$. **(4 marks)**

(b) Describe (without proof, but with careful attention to any special cases) the groups $H_n(\mathbb{R}^k \setminus \{0\})$ for all $n \geq 0$ and all $k \geq 1$. **(5 marks)**

(c) Let $u = n_1 s_1 + \ldots + n_k s_k$ be an m-cycle in S^n (where $m > 0$), and suppose that there is a point $a \in S^n$ that is not contained in any of the sets $s_1(\Delta_m), \ldots, s_k(\Delta_m)$. Prove that u is a boundary. **(8 marks)**

Solution:

(a) (i) The group $C_n(X)$ is the free Abelian group [1]generated by the set of continuous maps $s\colon \Delta_n \to X$ [1], where $\Delta_n = \{t \in \mathbb{R}^{n+1} \mid t_i \geq 0,\ \sum_i t_i = 1\}$. **[bookwork]**

(ii) We define continuous maps $\delta_0, \ldots, \delta_n\colon \Delta_{n-1} \to \Delta_n$ by

$$\delta_i(t_0, \ldots, t_{n-1}) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}).$$

For any continuous map $s\colon \Delta_n \to X$ we define

$$\partial_n(s) = \sum_{k=0}^{n} (-1)^k (s \circ \delta_k) \in C_{n-1}(X)[1].$$

This can be extended in a unique way to give a homomorphism $\partial_n\colon C_n(X) \to C_{n-1}(X)$. [1]**[bookwork]**

(iii) From the definitions, we have

$$\begin{aligned}\partial_1\partial_2[s] &= \partial_1([s\delta_0] - [s\delta_1] + [s\delta_2]) \\ &= [s\delta_0\delta_0] - [s\delta_0\delta_1] - [s\delta_1\delta_0] + [s\delta_1\delta_1] + [s\delta_2\delta_0] - [s\delta_2\delta_1] \\ &= ([s\delta_0\delta_0] - [s\delta_1\delta_0]) - ([s\delta_0\delta_1] - [s\delta_2\delta_0]) + ([s\delta_1\delta_1] - [s\delta_2\delta_1]).[1]\end{aligned}$$

Whenever $k \leq l$ we have $\delta_k\delta_l = \delta_{l+1}\delta_k$; this shows that each of the bracketed terms is zero [1]. Thus $\partial_2\partial_1$ vanishes on all singular 2-simplices, so it vanishes on all singular 2-chains [1]. **[bookwork]**

(iv) We define $Z_n(X) = \ker(\partial_n\colon C_n(X) \to C_{n-1}(X))$ [1]and $B_n(X) = \operatorname{img}(\partial_{n+1}\colon C_{n+1}(X) \to C_n(X))$ [1]. We have $\partial_n\partial_{n+1} = 0$, which implies that $B_n(X) \leq Z_n(X)$ [1], so we can define a quotient group $H_n(X) = Z_n(X)/B_n(X)$ [1]. **[bookwork]**

(b) As $\mathbb{R}^k \setminus \{0\}$ is homotopy equivalent to S^{k-1}, we have

$$H_n(\mathbb{R}^k \setminus \{0\}) = \begin{cases} \mathbb{Z}^2 & \text{if } n = 0, k = 1[2] \\ \mathbb{Z} & \text{if } n = 0, k > 1[1] \text{ or } n = k-1 > 0[1] \\ 0 & \text{otherwise } [1]. \end{cases}$$

[bookwork]

(c) The space $S^n \setminus \{a\}$ [2] is homeomorphic to $\mathbb{R}^n$ [1]by stereographic projection, and thus is contractible [1]. This implies that $H_m(S^n \setminus \{a\}) = 0$ for $m > 0$ [1], so every m-cycle in $S^n \setminus \{a\}$ is a boundary [1]. We can regard u as an m-cycle in $S^m \setminus \{a\}$, so it is a boundary in $S^n \setminus \{a\}$ [1]and thus in S^n [1], as required. **[unseen]**

(21)

(a) Let X be a topological space.

(i) Define the groups $C_0(X)$ and $C_1(X)$, and the homomorphism $\partial_1\colon C_1(X) \to C_0(X)$.

(ii) Define the subdivision homomorphism $\mathrm{sd}\colon C_1(X) \to C_1(X)$.

(iii) Prove that $\partial_1 \mathrm{sd}^n(u) = \partial_1(u)$ for all $n \geq 1$.

(iv) Prove that if $u \in B_1(X)$ then $\mathrm{sd}(u) \in B_1(X)$.

(v) Let A and B be points in a vector space V. Give an expression for $\mathrm{sd}\langle A, B\rangle$ in terms of paths of the form $\langle C, D\rangle$.

(b) Describe without proof the groups $H_1(S^1)$, $H_1(S^1 \times S^1)$, $H_1(\mathbb{R}P^2)$ and $H_1(\mathbb{R}^3 \setminus \{0\})$.

(c) For each element $u \in H_1(\mathbb{R}P^2)$, give a path s in $\mathbb{R}P^2$ such that $u = [s]$.

Solution:

(a) (i) $C_0(X) = \mathbb{Z}[X]$ is the free Abelian group on the set X, or in other words the group of all $\mathbb{Z}$-combinations of points of X. We also write $S_1(X)$ for the set of paths in X (in other words, continuous maps $s\colon \Delta_1 \to X$) and $C_1(X) = \mathbb{Z}[S_1X]$ for the group of 1-chains (in other words, $\mathbb{Z}$-combinations of paths). The homomorphism $\partial_1\colon C_1(X) \to C_0(X)$ is defined by

$$\partial_1(s) = s(e_1) - s(e_0),$$

extended linearly as usual.

(ii) Define maps $l, r\colon \Delta_1 \to \Delta_1$ by $l(1-t,t) = ((1+t)/2, (1-t)/2)$ and $r(1-t,t) = ((1-t)/2, (1+t)/2)$. The subdivision homomorphism $\mathrm{sd}\colon C_1(X) \to C_1(X)$ is defined by $\mathrm{sd}(s) = s \circ r - s \circ l$.

(iii) Given any path $s\colon \Delta_1 \to X$ we can define $q\colon \Delta_2 \to X$ by $q(t_0,t_1,t_2) = s(t_0/2 + t_1, t_0/2 + t_2)$. This satisfies

$$(q\delta_0)(1-t,t) = q(0, 1-t, t) = s(1-t,t)$$
$$(q\delta_1)(1-t,t) = q(1-t, 0, t) = s((1-t)/2, (1+t)/2) = sr(1-t,t)$$
$$(q\delta_2)(1-t,t) = q(1-t, t, 0) = s((1+t)/2, (1-t)/2) = sl(1-t,t).$$

Thus $\partial_2(q) = s - (s \circ r) + (s \circ (l) = s - \mathrm{sd}(s)$, showing that $s = \mathrm{sd}(s) \pmod{B_1(X)}$. By linear extension, we see that $u = \mathrm{sd}(u) \pmod{B_1(X)}$ for all $u \in C_1(X)$.

I claim that in fact $u \sim \mathrm{sd}^n(u)$ for all 1-chains u and all $n \geq 0$. The case $n = 0$ is clear because $\mathrm{sd}^0(u) = u$, and we have just done the case $n = 1$. Assume that the case $n = k-1$ holds. For any chain v we can apply the case $n = k-1$ to v to see that $v = \mathrm{sd}^{k-1}(v) \pmod{B_1(X)}$, and we can apply the case $n = 1$ to the chain $u = \mathrm{sd}^{k-1}(v)$ to see that $\mathrm{sd}^{k-1}(v) = \mathrm{sd}^k(v) \pmod{B_1(X)}$, and by putting these together we see that $v = \mathrm{sd}^k(v) \pmod{B_1(X)}$. By induction, this holds for all k.

(iv) Suppose that $u \in B_1(X)$, so $u = \partial(a)$ for some $a \in C_2(X)$. By part (iii) we have $u = \mathrm{sd}(u) \pmod{B_1(X)}$, in other words $\mathrm{sd}(u) - u = \partial(b)$ for some $b \in C_2(X)$. Thus $\mathrm{sd}(u) = \partial(a+b) \in B_1(X)$, as required.

(v) Put $C = (A+B)/2$ (the midpoint of the path $\langle A, B\rangle$). We have

$$\langle A,B\rangle(l(t)) = \langle A,B\rangle((1+t)/2, (1-t)/2) = ((1+t)/2)A + ((1-t)/2)B = (1-t)(A+B)/2 + tA$$
$$\langle A,B\rangle(r(t)) = \langle A,B\rangle((1-t)/2, (1+t)/2) = ((1-t)/2)A + ((1+t)/2)B = (1-t)(A+B)/2 + tB,$$

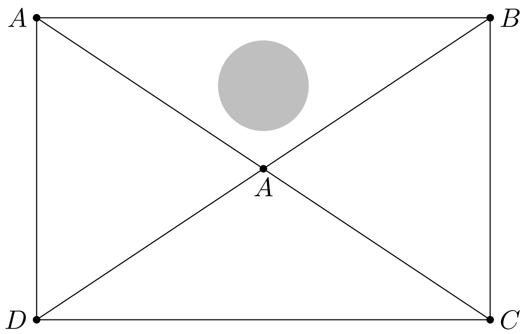
so $\langle A,B\rangle \circ l = \langle C,A\rangle)$ and $\langle A,B\rangle \circ r = \langle C,B\rangle$. Thus

$$\mathrm{sd}\langle A,B\rangle = \langle C,B\rangle - \langle C,A\rangle.$$

(b) $H_1(S^1) \simeq \mathbb{Z}$; $H_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$; $H_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2$; $H_1(\mathbb{R}^3 \setminus \{0\}) \simeq 0$.

(c) There are only two elements in $H_1(\mathbb{R}P^2)$, say 0 and v. For $u = 0$ we take s to be the constant path $s(1-t,t) = q(1,0,0)$. For $u = v$ we take $s(1-t,t) = q(\cos(\pi t), \sin(\pi t), 0)$.

(22) Consider the following diagram.



Let X be the complement in $\mathbb{R}^2$ of the shaded disc. Define $u, v, w \in C_1(X)$ by

$$u = \langle A,B\rangle + \langle B,E\rangle + \langle E,A\rangle$$
$$v = \langle A,B\rangle + \langle B,C\rangle + \langle C,D\rangle + \langle D,A\rangle$$
$$w = \langle A,E\rangle + \langle E,D\rangle + \langle D,A\rangle.$$

(a) Prove that u is a cycle. **(2 marks)**

(b) Prove that $\langle B,B\rangle$ is homologous to 0 in X. **(3 marks)**

(c) Prove that $\langle E,B\rangle$ is homologous to $-\langle B,E\rangle$ in X. **(4 marks)**

(d) Prove in detail that u is homologous to v in X, justifying each step. **(8 marks)**

(e) Write down a basic 1-chain s that is homologous in X to $\langle A, B\rangle + \langle B, C\rangle$ **(5 marks)**

(f) Is u homologous to w? Give a brief reason for your answer. **(3 marks)**

Solution:

(a) We have $\partial\langle U, V\rangle = V - U$ [1], so

$$\partial(u) = B - A + E - B + A - E = 0, [1]$$

so u is a cycle. **[similar examples seen]**

(b) We have $\partial\langle U, V, W\rangle = \langle V, W\rangle - \langle U, W\rangle + \langle U, V\rangle$ [1], so $\partial\langle B, B, B\rangle = \langle B, B\rangle - \langle B, B\rangle + \langle B, B\rangle = \langle B, B\rangle$ [1], which means that $\langle B, B\rangle$ is homologous to 0 [1]. **[bookwork]**

(c) We have $\partial\langle B, E, B\rangle = \langle E, B\rangle - \langle B, B\rangle + \langle B, E\rangle$ [2], so (using the previous part) we have

$$\langle E, B\rangle \sim \langle B, B\rangle - \langle B, E\rangle \sim -\langle B, E\rangle[2].$$

[similar examples seen]

(d) If we define $a = \langle A, D, E\rangle + \langle D, C, E\rangle + \langle C, B, E\rangle$ [3] we find that

$$\begin{aligned}\partial(a) =& \langle D, E\rangle - \langle A, E\rangle + \langle A, D\rangle + \\ & \langle C, E\rangle - \langle D, E\rangle + \langle D, C\rangle + \\ & \langle B, E\rangle - \langle C, E\rangle + \langle C, B\rangle \\ =& (\langle A, D\rangle + \langle D, C\rangle + \langle C, B\rangle) - \\ & (\langle A, E\rangle - \langle B, E\rangle).[3]\end{aligned}$$

Using this and part (iii), we see that

$$v \sim (\langle A, E\rangle - \langle B, E\rangle) \sim (\langle A, E\rangle + \langle E, B\rangle) = u.[2]$$

[similar examples seen]

(e) Define $s = \sigma(A, B) * \sigma(B, C)$, so

$$s(1-t, t) = \begin{cases} (1-2t)A + 2tB & \text{if } t \leq \frac{1}{2} \\ (2-2t)B + (2t-1)C & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Then $[s]$ is homologous to $\langle A, B\rangle + \langle B, C\rangle$ in X. [5]

(f) The chain u winds once around the hole, and w does not wind around the hole at all, so u is not homologous to w. [3]

7 True or false

(23) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) The punctured disc $X = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 1\}$ is compact.

(b) The circle S^1 is homeomorphic to $S^1 \times I$.

(c) The circle S^1 is homotopy equivalent to $S^1 \times I$.

(d) $\mathbb{C} \setminus S^1$ is homotopy equivalent to $Y = \{z \in \mathbb{C} \mid z = 0 \text{ or } |z| = 1\}$.

(e) Every continuous bijection from $[0, 1] \cup (2, 3]$ to $[0, 1]$ is a homeomorphism.

Solution:

(a) False. The space X is not closed in $\mathbb{R}^2$, because the sequence $(0, 1/n)$ in X converges in $\mathbb{R}^2$ to the point $(0,0)$, which does not lie in X. A subspace of $\mathbb{R}^n$ is compact iff it is bounded and closed, so X is not compact.

(b) False. Removing any two points disconnects S^1, but $S^1 \times I$ cannot be disconnected by removing any finite set of points.

(c) True. Define maps as follows:

$$\begin{aligned} &f\colon S^1 \to S^1 \times I && f(z) = (z, 0) \\ &g\colon S^1 \times I \to S^1 && g(z, r) = z \\ &h\colon I \times (S^1 \times I) \to S^1 \times I && h(t, (z, r)) = (z, tr). \end{aligned}$$

Then $gf = 1\colon S^1 \to S^1$ and h is a (linear) homotopy from fg to $1_{S^1 \times I}$, so f is a homotopy equivalence with homotopy inverse g.

(d) True. Define maps as follows:

$$\begin{aligned} &f\colon \mathbb{C} \setminus S^1 \to Y && f(z) = \begin{cases} z/|z| & \text{if } |z| > 1 \\ 0 & \text{if } |z| < 1 \end{cases} \\ &g\colon Y \to \mathbb{C} \setminus S^1 && g(z) = 2z. \end{aligned}$$

Then $fg = 1_Y$ and gf is linearly homotopic to $1_{\mathbb{C} \setminus S^1}$, so f is a homotopy equivalence with homotopy inverse g.

(e) False. Define $f\colon [0,1] \cup (2,3] \to [0,1]$ by

$$f(t) = \begin{cases} t/2 & \text{if } t \in [0,1] \\ (t-1)/2 & \text{if } t \in (2,3]. \end{cases}$$

Then f is a continuous bijection, but f^{-1} is not continuous (because $1/2 + 1/2n \to 1/2$ but $f^{-1}(1/2 + 1/2n) = 2 + 1/n$ does not converge to $f^{-1}(1/2) = 1$), so f is not a homeomorphism.

(24) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) S^3 is contractible.

(b) If a space X is the union of two closed, path-connected subspaces A and B, then X is path-connected.

(c) $(\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1.

(d) $(\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$ is homotopy equivalent to S^1.

(e) The space $\mathbb{C} \setminus \{0, 1\}$ is homeomorphic to $\mathbb{C} \setminus \{i, -i\}$.

(f) The space $\mathbb{C} \setminus \{0, 1\}$ is homotopy equivalent to $\mathbb{C} \setminus \{0, 1, 2\}$.

Solution:

(a) False. We have $H_3(S^3) = \mathbb{Z}$ but H_3 of a point is zero, so S^3 is not homotopy equivalent to a point.

(b) False. Put $X = \{0, 1\}$ and $A = \{0\}$ and $B = \{1\}$. Then A and B are closed path connected subsets of X with $X = A \cup B$, but X is not path connected. (You would not be required to say this, but I remark that if $X = A \cup B$ where A and B are path connected (not necessarily closed) and $A \cap B \neq \emptyset$ then X is path connected.)

(c) False. Write

$$X = (\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{R} \times \{0\}) = \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}.$$

We can then define a map $f\colon X \to \mathbb{R}$ by $f(x, y) = y$. This is never zero and it is positive at $(0, 1)$ and negative at $(0, -1)$, so $(0, 1)$ cannot be joined to $(0, -1)$ by a path in X, so X is not path connected. However, S^1 is path connected and anything homotopy equivalent to a path connected space is again path connected so X is not homotopy equivalent to S^1.

(d) True. Write $Y = (\mathbb{R} \times \mathbb{R}^2) \setminus (\mathbb{R} \times \{0\})$, and define maps as follows

$$\begin{aligned} f &: Y \to S^1 & f(x,y,z) &= (y,z)/\sqrt{y^2+z^2} \\ g &: S^1 \to Y & g(y,z) &= (0,y,z). \end{aligned}$$

Then $gf = 1_{S^1}$, and fg is linearly homotopic to 1_Y.

(e) True. We can define a homeomorphism $f\colon \mathbb{C} \setminus \{0,1\} \to \mathbb{C} \setminus \{i,-i\}$ by $f(z) = 2iz - i$, with inverse $f^{-1}(w) = (w+i)/2i$.

(f) False. We have $H_1(\mathbb{C} \setminus \{0,1\}) \simeq \mathbb{Z}^2$, and this is not isomorphic to $H_1(\mathbb{C} \setminus \{0,1,2\}) \simeq \mathbb{Z}^3$, so $\mathbb{C} \setminus \{0,1\}$ is not homotopy equivalent to $\mathbb{C} \setminus \{0,1,2\}$.

(25) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) The identity map of the unit circle is homotopic to the constant map $c\colon S^1 \to S^1$ defined by $c(z) = 1$ for all z.

(b) Let $f_n\colon S^1 \to S^1$ be defined by $f_n(z) = z^n$. Then f_n is not homotopic to f_m when $n \neq m$.

(c) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$.

(d) If $f\colon X \to X$ is a homotopy equivalence, then $f_*\colon H_1(X) \to H_1(X)$ is the identity map.

Solution:

(a) False. Define $s_1\colon \Delta_1 \to S^1$ by $s_1(t) = e(t)$, so that $u_1 = ([s_1] \bmod B_1(S^1))$ is the usual generator of $H_1(S^1)$. Then $c \circ s_1\colon \Delta_1 \to S^1$ is a constant path, so $c_*[s_1] \sim 0$, so $c_*(u_1) = 0$ in $H_1(S^1)$. Thus c_* is not the identity map on $H_1(S^1)$, so c is not homotopic to the identity map on S^1.

(b) True. Define $s_n = f_n \circ s_1\colon \Delta_1 \to S^1$, so $s_n(t) = e(t)^n = e(nt)$, so s_n can be unwound to the path $\tilde{s}_n(t) = nt$ in $\mathbb{R}$. It follows that the usual isomorphism $\overline{\phi}\colon H_1(S^1) \to \mathbb{Z}$ satisfies

$$\overline{\phi}([s_n] \bmod B_1(S^1)) = \tilde{s}_n(1) - \tilde{s}_n(0) = n = \overline{\phi}(nu_1),$$

so $f_{n*}(u_1) = nu_1$. It follows that $f_{n*} \neq f_{m_*}$ when $n \neq m$, so f_n is not homotopic to f_m when $n \neq m$.

(c) False. If $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ were a homeomorphism, then it would give a homeomorphism $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{f(0)\}$. However, $\mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to S^1 and $\mathbb{R}^3 \setminus \{f(0)\}$ is homeomorphic to $\mathbb{R}^3 \setminus \{0\}$ and thus homotopy equivalent to S^2. We know that $H_1(S^1) \simeq \mathbb{Z}$ and $H_1(S^2) \simeq 0$ so S^1 is not homotopy equivalent to S^2. It follows that $\mathbb{R}^2 \setminus \{0\}$ is not homotopy equivalent (and thus certainly not homeomorphic) to $\mathbb{R}^3 \setminus \{f(0)\}$, so no such map f can exist.

(d) False. The map $f_{-1}\colon S^1 \to S^1$ is a homeomorphism and thus a homotopy equivalence, and $(f_{-1})_*(u_1) = -u_1$ so $(f_{-1})_*$ is not the identity map.

(26) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems, provided that you state them clearly.

(a) The torus $T = S^1 \times S^1$ is homotopy equivalent to S^2.

(b) There is a map $r\colon B^4 \to S^3$ such that rj is homotopic to id_{S^3}, where $j\colon S^3 \to B^4$ is the inclusion map.

(c) $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^3$.

(d) Every continuous function $f\colon S^2 \to \mathbb{R}^3$ is homotopic to a constant function.

(e) Let $K \subset S^3$ be a trefoil knot. Then $S^3 \setminus K$ is homotopy equivalent to $\mathbb{R}P^2$.

Solution:

(a) False. We have $H_1(T) \simeq \mathbb{Z}^2$, but $H_1(S^2) = 0$, so T is not homotopy equivalent to S^2.

(b) False. Let u_3 be the usual generator of H_3S^3. If there were such a map r, we would have $r_*j_* = (rj)_* = 1_* = 1\colon H_3S^3 \to H_3S^3$, so $u_3 = r_*(j_*(u_3))$. This is impossible, because B^4 is contractible so $H_3B^4 = 0$ and $j_*(u_3) \in H_3B^4$ so $j_*(u_3) = 0$ so $r_*(j_*(u_3)) = 0$.

(c) False. If $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ were a homeomorphism, then it would give a homeomorphism $\mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{f(0)\}$. However, $\mathbb{R}^2 \setminus \{0\}$ is homotopy equivalent to S^1 and $\mathbb{R}^3 \setminus \{f(0)\}$ is homeomorphic to $\mathbb{R}^3 \setminus \{0\}$ and thus homotopy equivalent to S^2. We know that $H_1(S^1) \simeq \mathbb{Z}$ and $H_1(S^2) \simeq 0$ so S^1 is not homotopy equivalent to S^2. It follows that $\mathbb{R}^2 \setminus \{0\}$ is not homotopy equivalent (and thus certainly not homeomorphic) to $\mathbb{R}^3 \setminus \{f(0)\}$, so no such map f can exist.

(d) True. We can just define $h(t, x) = tf(x)$; this is a homotopy from the constant map with value 0 to f.

(e) False. We have $H_1(S^3 \setminus K) \simeq \mathbb{Z}$ by the generalised Jordan Curve Theorem, but $H_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2$.

(27) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems or calculations of homology groups without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) $\mathbb{R}P^1$ is homeomorphic to S^1.

(b) The Möbius strip is homotopy equivalent to S^2.

(c) $S^2 \setminus S^1$ is homotopy equivalent to $\mathbb{R} \setminus \{0\}$.

(d) The letter A is homeomorphic to the letter D.

(e) Any compact convex subset of $\mathbb{R}^2$ is homeomorphic to B^2.

Solution:

(a) True. There is a homeomorphism $f\colon S^1 \to \mathbb{R}P^1$ given by

$$f(x, y) = \frac{1}{2}\begin{pmatrix} 1 + x & y \\ y & 1 - x \end{pmatrix}.$$

(b) This is false, because the Möbius strip M is homotopy equivalent to S^1, so $\pi_1(M) \simeq \pi_1(S^1) \simeq \mathbb{Z}$, whereas $\pi_1(S^2) = 0$.

(c) This is true because both spaces are homotopy equivalent to the space with two points. Indeed, $\mathbb{R} \setminus \{0\}$ is the disjoint union of two contractible spaces $(-\infty, 0)$ and $(0, \infty)$, each of which is homotopy equivalent to a point, so $\mathbb{R} \setminus \{0\}$ is homotopy equivalent to two points. Similarly, $S^2 \setminus S^1$ is the disjoint union of the sets $U_+ = \{(x, y, z) \in S^2 \mid z > 0\}$ and $U_- = \{(x, y, z) \in S^2 \mid z < 0\}$. If we put $V = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ then V is contractible and there is a homeomorphism $f_+\colon V \to U_+$ given by $f(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$, so U_+ is contractible. Similarly U_- is contractible, so $S^2 \setminus S^1$ is again homotopy equivalent to two points.

(d) This is false, because A can be disconnected by removing a point in the middle of one of the legs, but D cannot be disconnected by removing a single point.

(e) This is false: the closed line segment from $(-1, 0)$ to $(1, 0)$ is compact and convex but not homeomorphic to B^2. (The theorem states that if $X \subseteq \mathbb{R}^2$ is compact and convex *and contains an open ball* then X is homeomorphic to B^2.)

(28) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems or calculations of homology groups without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) $\mathbb{R}P^1$ is homeomorphic to S^1.

(b) The Möbius strip is homotopy equivalent to S^2.

(c) $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.

(d) $S^2 \setminus S^1$ is homotopy equivalent to $\mathbb{R} \setminus \{0\}$.

(e) The letter A is homeomorphic to the letter D.

(f) Any compact convex subset of $\mathbb{R}^2$ is homeomorphic to B^2.

Solution:

(a) True. If we regard S^1 as the unit circle in the complex plane then we have $z \sim w$ iff $z^2 = w^2$, so there is a well-defined function $f: \mathbb{R}P^1 \to S^1$ given by $f(q(z)) = z^2$, and this is a homeomorphism.

(b) This is false, because the Möbius strip M is homotopy equivalent to S^1, so $H_1(M) \simeq H_1(S^1) \simeq \mathbb{Z}$, whereas $H_1(S^2) = 0$.

(c) This is true by a formula given in the notes, but it turns out that I won't have time to explain this properly.

(d) This is true because both spaces are homotopy equivalent to the space with two points. Indeed, $\mathbb{R} \setminus \{0\}$ is the disjoint union of two contractible spaces $(-\infty, 0)$ and $(0, \infty)$, each of which is homotopy equivalent to a point, so $\mathbb{R} \setminus \{0\}$ is homotopy equivalent to two points. Similarly, $S^2 \setminus S^1$ is the disjoint union of the sets $U_+ = \{(x, y, z) \in S^2 \mid z > 0\}$ and $U_- = \{(x, y, z) \in S^2 \mid z < 0\}$. If we put $V = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ then V is contractible and there is a homeomorphism $f_+: V \to U_+$ given by $f(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$, so U_+ is contractible. Similarly U_- is contractible, so $S^2 \setminus S^1$ is again homotopy equivalent to two points.

(e) This is false, because A can be disconnected by removing a point in the middle of one of the legs, but D cannot be disconnected by removing a single point.

(f) This is false: the closed line segment from $(-1, 0)$ to $(1, 0)$ is compact and convex but not homeomorphic to B^2. (The theorem states that if $X \subseteq \mathbb{R}^2$ is compact and convex *and contains an open ball* then X is homeomorphic to B^2.)

(29) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems or calculations of homology groups without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) S^1 is homotopy equivalent to S^2. **(3 marks)**

(b) S^1 is homotopy equivalent to the Möbius strip. **(4 marks)**

(c) S^1 is homeomorphic to the Möbius strip. **(4 marks)**

(d) $\mathbb{R}P^2$ is homeomorphic to $S^1 \times S^1$. **(4 marks)**

(e) $SU(2) \setminus \{I\}$ is homeomorphic to $\mathbb{R}^3$. **(5 marks)**

(f) $\Delta_n \times \Delta_m$ is homeomorphic to Δ_{n+m}. **(5 marks)**

Solution:

(a) False. We have $H_1(S^1) \simeq \mathbb{Z}$ [1]but $H_1(S^2) = 0$ [1]. If X is homotopy equivalent to Y, then $H_n(X) \simeq H_n Y$ for all n [1], so we conclude that $S^1 \not\simeq S^2$. **[seen]**

(b) True. The Möbius strip M is the quotient of $\mathbb{R} \times [-1, 1]$ by the equivalence relation

$$(x, y) \sim (x', y') \text{ iff } (x - x' \in \mathbb{Z} \text{ and } y = (-1)^{x-x'} y').$$

The circle can be thought of as the quotient of $\mathbb{R}$ by the equivalence relation

$$x \sim x' \text{ iff } x - x' \in \mathbb{Z}.$$

We thus have a map $j: S^1 \to M$ defined by $j\langle x \rangle = \langle x, 0 \rangle$ [1], and a map $q: M \to S^1$ defined by $q\langle x, y \rangle = \langle x \rangle$ [1]; these clearly satisfy $qj = 1$. We also have a map $h: I \times M \to M$ defined by $h(t, \langle x, y \rangle) = \langle x, ty \rangle$ [1]. This has $h(1, \langle x, y \rangle) = \langle x, y \rangle$, and $h(0, \langle x, y \rangle) = \langle x, 0 \rangle = jq\langle x, y \rangle$, so that $jq \simeq 1$. [1]**[seen]**

(c) False. If we remove any two distinct points from S^1 it becomes disconnected, but this is clearly not true for M. [4] **[similar examples seen]**

(d) We calculated in lectures that $H_1(\mathbb{R}P^2) \simeq \mathbb{Z}/2$ [1], whereas $H_1(S^1 \times S^1) \simeq \mathbb{Z} \times \mathbb{Z}$ [2]. It follows as in (a) that $\mathbb{R}P^2$ is not homotopy equivalent (and so not homeomorphic [1]) to $S^1 \times S^1$. **[similar examples seen]**

(e) True. There is a homeomorphism $f\colon S^3 \to SU(2)$ given by

$$f(a,b,c,d) = \begin{pmatrix} a+ib & c+id \\ c-id & a-ib \end{pmatrix}.[2]$$

If we define $P = (1,0,0,0)$ then $f(P) = I$, so f induces a homeomorphism $S^3 \setminus \{P\} \to SU(2) \setminus \{I\}$. On the other hand, stereographic projection gives a homeomorphism $g\colon S^3 \setminus \{P\} \to \mathbb{R}^3$. Explicitly, we have

$$g(a,b,c,d) = (b/(1-a), c/(1-a), d/(1-a)).[3]$$

(f) True. We proved in lectures that if $X \subseteq \mathbb{R}^k$ is bounded, closed and convex and contains an open ball, then X is homeomorphic to B^k [3]. This applies to both the sets $\Delta_n \times \Delta_m$ and Δ_{n+m}, so $\Delta_n \times \Delta_m \simeq B^{n+m} \simeq \Delta_{n+m}$ [2]. **[similar examples seen]**

(30) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) There is a continuous surjective map from $S^1 \times S^1$ to $\mathbb{R}$

(b) $\mathbb{C} \setminus \{2\}$ is homotopy equivalent to S^1

(c) $\mathbb{C} \setminus \{-1, 1\}$ is homotopy equivalent to S^1

(d) $S^2 \setminus \{\text{the north pole}\}$ is homeomorphic to $\mathbb{C}$.

(e) The letter X (considered as a subspace of $\mathbb{R}^2$) is homeomorphic to the letter Y.

(f) The letter X (considered as a subspace of $\mathbb{R}^2$) is homotopy equivalent to the letter Y.

Solution:

(a) **False.** The space $S^1 \times S^1$ is compact, and $\mathbb{R}$ is not compact, so there can be no continuous surjection from $S^1 \times S^1$ to $\mathbb{R}$.

(b) **True.** The map $f\colon \mathbb{C} \setminus \{2\} \to S^1$ given by $f(z) = (z-2)/|z-2|$ is a homotopy equivalence.

(c) **False.** The space $\mathbb{C} \setminus \{-1, 1\}$ is homotopy equivalent to the figure eight, so its fundamental group is nonabelian, whereas $\pi_1(S^1)$ is isomorphic to $\mathbb{Z}$ and thus is abelian. This shows that the two spaces have non-isomorphic fundamental groups, so they cannot be homotopy equivalent.

(d) **True.** The space $S^2 \setminus \{\text{the north pole}\}$ is homeomorphic to $\mathbb{R}^2$ by stereographic projection, and of course $\mathbb{R}^2$ is homeomorphic to $\mathbb{C}$ by the correspondence $(x, y) \leftrightarrow x + iy$.

(e) **False.** We can remove the central point from the letter X and the resulting space has four path components; but if we remove a point from the letter Y, the remaining space has at most three path components. This shows that X is not homeomorphic to Y.

(f) **True.** The letter X is star-shaped around its central point, so it is contractible, and the same applies to Y. Thus, they are both homotopy equivalent to a point and hence to each other.

(31) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) There is a continuous surjective map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \setminus \{0\}$ **(5 marks)**

(b) $S^2 \setminus S^1$ is homeomorphic to $\mathbb{R}^2$ **(5 marks)**

(c) $SO(2)$ is homotopy equivalent to the Möbius strip **(5 marks)**

(d) $SO(3)$ is homotopy equivalent to the torus **(5 marks)**

(e) The space $X = S^1 \cup \{(x, 0) \mid x \in \mathbb{R}\}$ is homeomorphic to $Y = S^1 \cup \{(x, 1) \mid x \in \mathbb{R}\}$. **(5 marks)**

Solution: In each part, two marks will be awarded for a correct true/false answer with no justification, and up to three (or exceptionally four) marks may be awarded for a reasonable line of argument leading to the wrong answer.

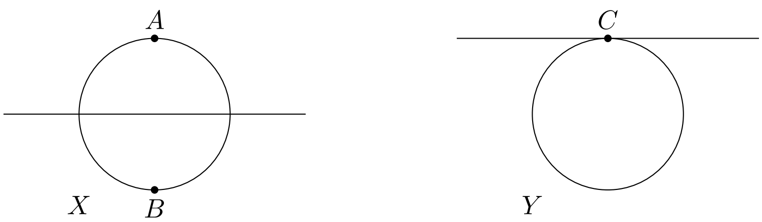
(a) False. The space $\mathbb{R}^2$ is path-connected and $\mathbb{R} \setminus \{0\}$ is not, so there cannot be a continuous surjective map from $\mathbb{R}^2$ to $\mathbb{R} \setminus \{0\}$. [5] **[similar examples seen]**

(b) False. The space $S^2 \setminus S^1$ is homotopy equivalent to S^0, and thus is not path-connected; but $\mathbb{R}^2$ is evidently path-connected, by linear paths. [5] **[seen]The homotopy equivalence $S^n \setminus S^m \simeq S^{n-m-1}$ is in the summary, so I expect that the students will use it. There are of course more direct proofs that $S^2 \setminus S^1$ is disconnected; they are also acceptable.**

(c) True. $SO(2)$ is homeomorphic to S^1 **[seen]**, and the Möbius strip M is homotopy equivalent to the circle running along the middle of the strip **[seen]**, so $SO(2)$ is homotopy equivalent to M. [5] **The summary contains various lists of spaces that are all homotopy equivalent to each other; one such list contains S^1, $SO(2)$ and the Möbius strip.**

(d) False. We know that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$, so $H_1(SO(3))$ has order two. On the other hand, $H_1(T) \simeq \mathbb{Z} \times \mathbb{Z}$ is infinite, and thus not isomorphic to $H_1(SO(3))$, so T is not homotopy equivalent to $SO(3)$. [5] **[similar examples seen]The facts that $SO(3) \simeq \mathbb{R}P^3$, that $H_1(\mathbb{R}P^3) = \mathbb{Z}/2$, and that $H_1(T) = \mathbb{Z} \times \mathbb{Z}$ are all in the summary. It is mentioned explicitly in the notes that $H_1(SO(3)) = \mathbb{Z}/2$. Many examples are given where we use H_1 to show that two spaces are not homotopy equivalent.**

(e) False. The picture is as follows:



The points A and B can be removed from X without disconnecting it; but removing any two points from Y disconnects it. (Note in particular that removing C already disconnects Y). [5] **[similar examples seen]**

(32) Are the following statements true or false? Give proof or disproof as appropriate. You may quote general theorems without proof, provided that you state them clearly. You may give pictures instead of formulae, provided that they are clearly explained.

(a) If X and Y are both path-connected subsets of $\mathbb{R}^2$, then $X \cap Y$ is also path-connected. **(5 marks)**

(b) The torus is homotopy equivalent to S^2. **(5 marks)**

(c) If $f : X \to Y$ and $g : Y \to X$ are continuous, based maps and $gf = \mathrm{id}_X$ then $\pi_1(X) \simeq \pi_1(Y)$. **(5 marks)**

(d) If two letters of the alphabet, considered as subspaces of $\mathbb{R}^2$, both have infinite H_1, then they are homotopy equivalent. **(5 marks)**

(e) The space $GL_3(\mathbb{R})$ is path-connected. **(5 marks)**

Solution: In each part, two marks will be awarded for a correct true/false answer with no justification, and up to three (or exceptionally four) marks may be awarded for a reasonable line of argument leading to the wrong answer.

(a) False. If $X = \{(x, y) \in S^1 \mid y \geq 0\}$ and $Y = \{(x, y) \in S^1 \mid y \leq 0\}$ then X and Y are path-connected but $X \cap Y$ is not.

(b) False. We have $\pi_1(T) = \mathbb{Z} \times \mathbb{Z}$ and $\pi_1(S^2) = 0$ so T cannot be homotopy equivalent to S^2.

(c) False. Take $X = \{0\}$, $Y = S^1$, $f(0) = 1$ and $g(x, y) = 0$. Then $gf = \mathrm{id}_X$ but $\pi_1(X) = 0$ whereas $\pi_1(Y) = \mathbb{Z}$.

(d) False. The letter O has $H_1(O) = \mathbb{Z}$, whereas B has $H_1(B) = \mathbb{Z}^2$. These two homology groups are both infinite but are not isomorphic, so the spaces are not homotopy equivalent.

(e) False. The determinant gives a continuous map $\det\colon GL_3(\mathbb{R}) \to \mathbb{R} \setminus \{0\}$, which takes a positive value at I and a negative value at $-I$, so there can be no path from I to $-I$ in $GL_3(\mathbb{R})$.

(33) 2018-19 Q2: Are the following true or false? Justify your answers.

(a) S^5 is a Hausdorff space. **(4 marks)**

(b) The Klein bottle is a retract of $S^1 \times S^1 \times S^1$. **(4 marks)**

(c) There is a connected space X with $\pi_1(X) \simeq \mathbb{Z}/2$ and $H_1(X) \simeq \mathbb{Z}$. **(4 marks)**

(d) There is a short exact sequence $\mathbb{Z}/9 \to \mathbb{Z}/99 \to \mathbb{Z}/11$. **(4 marks)**

(e) If K is a simplicial complex and L is a subcomplex and $H_3(K) = 0$ then $H_3(L) = 0$. **(4 marks)**

(f) If K and L are simplicial complexes and $f\colon |K| \to |L|$ is a continuous map then there is a simplicial map $s\colon K \to L$ such that f is homotopic to $|s|$. **(5 marks)**

Solution:

(a) This is true [1], because the standard topology on S^5 comes from the Euclidean metric on $\mathbb{R}^6$, and metric spaces are always Hausdorff. [3] **[It was proved in lectures that metric spaces are Hausdorff.]**

(b) This is false [1]. Let X be the Klein bottle. If this was a retract of $(S^1)^3$, then $\pi_1(X)$ would be a retract of the group $\pi_1((S^1)^3) = \mathbb{Z}^3$, so in particular it would be a subgroup of $\mathbb{Z}^3$ and so would be abelian. However, it is standard that $\pi_1(X)$ is nonabelian, so this is a contradiction. [3] **[Similar examples have been seen.]**

(c) This is false [1]. For a connected space X, the group $H_1(X)$ is always the abelianisation of $\pi_1(X)$. Thus, if $\pi_1(X)$ is $\mathbb{Z}/2$ then $H_1(X)$ must also be $\mathbb{Z}/2$. [3] **[Unseen]**

(d) This is true [1]: there is a short exact sequence $\mathbb{Z}/9 \xrightarrow{i} \mathbb{Z}/99 \xrightarrow{p} \mathbb{Z}/11$ given by $i(a \pmod 9) = 11a \pmod{99}$ and $p(b \pmod{99}) = b \pmod{11}$. [3] Alternatively, as 9 and 11 are coprime we can use the Chinese Remainder Theorem to identify $\mathbb{Z}/99$ with $\mathbb{Z}/9 \times \mathbb{Z}/11$. We then have a short exact sequence $\mathbb{Z}/9 \xrightarrow{j} \mathbb{Z}/9 \times \mathbb{Z}/11 \xrightarrow{q} \mathbb{Z}/11$ given by $j(x) = (x, 0)$ and $q(x, y) = y$. **[Similar examples have been seen.]**

(e) This is false [1]. For example, if $K = \Delta^4$ and $L = \partial\Delta^4 \subset K$ then $H_3(K) = 0$ but $H_3(L) = \mathbb{Z}$. [3] **[Seen]**

(f) This is false. [1]For example, K and L could be as follows:

If $s\colon K \to L$ is a simplicial map, it is easy to see that the image can only be a single point or a single edge of L, and thus that $|s|$ is homotopic to a constant map. However, it is easy to produce a homeomorphism $f\colon |K| \to |L|$ and then f is not homotopic to a constant, so it cannot be homotopic to $|s|$ for any s. [4] (By the Simplicial Approximation Theorem, for any $f\colon |K| \to |L|$ we can find a corresponding map $s\colon K^{(r)} \to L$ for sufficiently large r; but that is not relevant here, because the question specifies that s should be defined on K itself.) **[Similar examples have been seen.]**

8 Examples

(34) Give examples of the following things, with careful justification.

(a) A noncompact metric space X with a sequence of compact subspaces $Y_1 \subset Y_2 \subset \dots$ such that the union of all the sets Y_n is equal to X.

(b) A metric space X with two noncompact subsets Y, Z such that $Y \cup Z$ is compact.

(c) A sequence in $\mathbb{R}$ with no convergent subsequence.

(d) A non-surjective map $f\colon X \to Y$ such that $f_*\colon \pi_0(X) \to \pi_0(Y)$ is surjective.

(e) An injective map $f\colon X \to Y$ such that $f_*\colon \pi_0(X) \to \pi_0(Y)$ is not injective.

Solution:

(a) Take $X = \mathbb{R}$, $Y_n = [-n, n]$. The sequence $(1, 2, 3, \dots)$ in $\mathbb{R}$ has no convergent subsequence, so $\mathbb{R}$ is noncompact. Moreover, Y_n is a bounded closed subspace of $\mathbb{R}$ and thus is compact. For any $x \in \mathbb{R}$ we can choose an integer $n > |x|$ and then $x \in Y_n$, which shows that $X = Y_1 \cup Y_2 \cup \dots$.

(b) Put $X = [0, 1]$ and $Y = (0, 1]$ and $Z = [0, 1)$. Then $Y \cup Z = [0, 1]$ which is compact. The sequence $(1/n)$ in Y has n subsequence that converges in Y, so Y is noncompact. Similarly, the sequence $(1 - 1/n)$ in Z has no subsequence that converges in Z, so Z is noncompact.

(c) The sequence $1, 2, 3, \dots$ in $\mathbb{R}$ has no convergent subsequence, because any two distict terms have distance at least one apart so no subsequence can be Cauchy.

(d) Let $X = \{0\}$ and $Y = [0, 1]$ and define $f\colon X \to Y$ by $f(0) = 0$. Then X and Y are both path-connected, so $\pi_0(X)$ and $\pi_0(Y)$ have only one point each. The map $f_*\colon \pi_0(X) \to \pi_0(Y)$ sends the only point in $\pi_0(X)$ to the only point in $\pi_0(Y)$, so f_* is a bijection and in particular is surjective. However f is obviously not surjective, as 1 does not lie in the image of f for example.

(e) Put $X = \{0, 1\}$ and $Y = [0, 1]$ and let $f\colon X \to Y$ be the inclusion map, which is clearly injective. If we write a for the component of 0 in X and b for the component of 1 in X and c for the component of 0 in Y then $\pi_0(X) = \{a, b\}$ and $\pi_0(Y) = \{c\}$ and $f_*(a) = f_*(b) = c$, so f_* is not injective.

(35) Give examples of the following things, with careful justification.

(a) A continuous bijection that is not a homeomorphism. **(3 marks)**

(b) An infinite sequence of open sets whose intersection is not open. **(3 marks)**

(c) Two metric spaces X, Y such that X is bounded, Y is unbounded, and X is homeomorphic to Y. **(4 marks)**

(d) A sequence in $(0, 1)$ such that no subsequence converges in $(0, 1)$. **(5 marks)**

(e) Two contractible subsets of $\mathbb{R}^2$ whose intersection is not contractible. **(5 marks)**

(f) Two metric spaces X, Y and points $x \in X$, $y \in Y$ such that X is homotopy equivalent to Y but $X \setminus \{x\}$ is not homotopy equivalent to $Y \setminus \{y\}$. **(5 marks)**

Solution:

(a) Put $X = ([-1, 0] \times \{0\}) \cup ((0, 1] \times \{1\}) \subset \mathbb{R}^2$, and $Y = [-1, 1] \subset \mathbb{R}$ [1]. The map $q\colon X \to Y$ defined by $q(x, y) = x$ [1]is a continuous bijection, but not a homeomorphism (because Y is sequentially compact and X is not, for example) [1]. **[seen]**

(b) Put $U_n = (-1/n, 1/n)$ [2], which is open in $\mathbb{R}$. The intersection of all the sets U_n is the one-point set $\{0\}$, [1]which is not open in $\mathbb{R}$. **[seen]**

(c) Put $X = (0, 1)$ and $Y = (1, \infty)$, so clearly X is bounded and Y is not. Define $f\colon X \to Y$ by $f(x) = 1/x$. This is a homeomorphism, with $f^{-1}(y) = 1/y$. [4] **[seen]**

(d) Put $x_n = 1/n$ [2]. This converges in $\mathbb{R}$ to 0, so any subsequence converges in $\mathbb{R}$ to 0 [2], so it has no limit in $(0, 1)$. [1][**seen**]

(e) Put

$$X_+ = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \text{ and } y \geq 0\}$$
$$X_- = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \text{ and } y \leq 0\}.\text{[2]}$$

Then X_+ and X_- are homeomorphic to I and thus are contractible [1], but $X_+ \cap X_- = \{(-1, 0), (1, 0)\}$ is not path-connected [1]and thus certainly not contractible [1]. [**unseen**]

(f) Take $X = Y = I$ and $x = 0$ and $y = 1/2$ [3]. Then X and Y are contractible, and thus certainly homotopy equivalent to each other [1]. However, $X \setminus \{x\}$ is path-connected and $Y \setminus \{y\}$ is not, so $X \setminus \{x\}$ is not homotopy equivalent to $Y \setminus \{y\}$ [1]. [**unseen**]

(36) Give examples of the following things, with justification.

(a) Connected sets $X, Y \subseteq \mathbb{R}^2$ such that $X \cap Y$ is not connected.

(b) A sequence of open sets $U_n \subseteq \mathbb{R}$ such that the set $X = U_1 \cap U_2 \cap \ldots = \bigcap_n U_n$ is not open.

(c) A surjective map $f \colon X \to Y$ of topological spaces such that the homomorphism $f_* \colon H_1(X) \to H_1(Y)$ is not surjective.

(d) A path connected space X that is homotopy equivalent to $X \times X$.

(e) A path connected space X that is not homotopy equivalent to $X \times X$.

Solution:

(a) Put

$$\begin{aligned} X &= \text{the upper half of the unit circle} \\ &= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\ ,\ y \geq 0\} \\ Y &= \text{the lower half of the unit circle} \\ &= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\ ,\ y \leq 0\} \end{aligned}$$

Then X and Y are both connected, but $X \cap Y = \{(-1, 0), (1, 0)\}$, which is disconnected.

(b) Put $U_n = \{x \in \mathbb{R} \mid |x| < 1/n\} = (-1/n, 1/n)$. Then U_n is open in $\mathbb{R}$, but

$$\begin{aligned} \bigcap_n U_n &= \{x \in \mathbb{R} \mid |x| < 1/n \text{ for all } n\} \\ &= \{x \in \mathbb{R} \mid |x| = 0\} \\ &= \{0\}, \end{aligned}$$

which is not open.

(c) Define $\eta \colon \mathbb{R} \to S^1$ by $\eta(t) = \exp(2\pi i t)$, which gives a surjective, continuous map. As $H_1(\mathbb{R}) = \{e\}$ and $H_1(S^1)$ is infinite, it is clear that $\eta_* \colon H_1(\mathbb{R}) \to H_1(S^1)$ cannot be surjective.

(d) The spaces I and $I \times I$ are both homotopy equivalent to a point, and thus to each other. (For a more degenerate example, one could just take X to be a point.)

(e) The space S^1 is not homotopy equivalent to $S^1 \times S^1$ (because $H_1(S^1) = \mathbb{Z}$ is not isomorphic to $H_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$).

(37) Give examples of the following things.

(a) A space X and a point $x \in X$ such that X is not contractible but $X \setminus \{x\}$ is contractible. **(3 marks)**

(b) A subspace $X \subseteq \mathbb{R}^2$ that is homotopy equivalent to $S^4 \setminus S^2$. (You need not give a proof.) **(4 marks)**

(c) Spaces X and Y, a discontinuous map $f\colon X \to Y$, and an open subset $V \subseteq Y$ such that $f^{-1}V$ is not open in X. (You should justify your answer carefully.) **(6 marks)**

(d) A space X and a point $x \in X$ such that $\pi_1(X)$ is abelian and $\pi_1(X \setminus \{x\})$ is nonabelian. (You should state what $\pi_1(X)$ and $\pi_1(X \setminus \{x\})$ are, but no further justification is required.) **(6 marks)**

(e) A space X such that $a(X) = 2$ and $b(X) = 2$, where as usual

$$
\begin{aligned}
a(X) &= \max\{|S| \mid S \text{ is a finite subset of } X \text{ and } X \setminus S \text{ is path-connected }\} \\
&= \text{the largest number of points that can be} \\
& \text{removed from } X \text{ without disconnecting it} \\
b(X) &= \min\{|S| \mid S \text{ is a finite subset of } X \text{ and } X \setminus S \text{ is not path-connected }\} \\
&= \text{the smallest number of points that have to be} \\
& \text{removed from } X \text{ to disconnect it}
\end{aligned}
$$

(You should justify your answer, but complete rigour is not required.) **(6 marks)**

Solution:

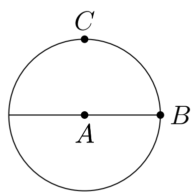
(a) S^1 is not contractible (because $H_1(S^1) = \mathbb{Z}$ is nontrivial) but $S^1 \setminus \{1\}$ is homeomorphic to $\mathbb{R}$ and thus is contractible. [3] **[seen]These facts are in the summary.**

(b) In general, $S^n \setminus S^m$ is homotopy equivalent to S^{n-m-1}. In particular, the space $S^4 \setminus S^2$ is homotopy equivalent to S^1, which is a subset of $\mathbb{R}^2$ [4]. **[seen]The homotopy equivalence $S^n \setminus S^m \simeq S^{n-m-1}$ is in the summary.**

(c) Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = 0$ for $x \leq 0$ and $f(x) = 1$ for $x > 0$. [1]This is discontinuous at $x = 0$, [1]because $1/n \to 0$ but $f(1/n) = 1 \not\to 0 = f(0)$ [1]. If we put $V = (-1, 1) \subset \mathbb{R}$ [1]then $f^{-1}V = (-\infty, 0]$ [1]. Thus V is open but $f^{-1}V$ is not [1]. **[seen]**

(d) Put $X = T = S^1 \times S^1$ and $x = (1, 1)$ [2]. Then $\pi_1(X) = \mathbb{Z} \times \mathbb{Z}$ [1], which is abelian. However, $X \setminus \{x\}$ is homotopy equivalent to a figure eight [1], so $\pi_1(X \setminus \{x\})$ is the free group on two generators [1], which is not abelian [1]. **[seen]These facts are in the summary. The only spaces ever mentioned with nonabelian fundamental group are the figure eight, the torus with one puncture, and the plane with two punctures. The fact that these three spaces are homotopy equivalent is also in the summary.**

(e) We can take X to be the letter B, or the following space, which is homeomorphic to the letter B: [3]



It is clear that $X \setminus \{A, C\}$ is connected (so $a(X) \geq 2$) and $X \setminus \{A, B\}$ is not (so $b(X) \leq 2$). By inspection, if we remove any one point, then X remains connected, so $b(X) = 2$. Also, if we remove any three points, then X becomes disconnected, so $a(X) = 2$. [3] **[seen]The calculation of a(letter B) and b(letter B) has been seen (and similarly for various other letters, and some other spaces). However, the students have not previously been asked to find a space with prescribed values of $a(X)$ and $b(X)$.**

(38) Give one example of each of the following things, with justification.

(a) A path connected space X with $H_1(X) = \mathbb{Z} \oplus (\mathbb{Z}/2)$. **(4 marks)**

(b) A path-connected space X and points $a, b, c \in X$ such that $X \setminus \{a, b, c\}$ is still path-connected. **(3 marks)**

(c) A path-connected space X and a point $a \in X$ such that $H_1(X)$ and $H_1(X \setminus \{a\})$ are both trivial. **(5 marks)**

(d) A continuous, surjective map $f\colon X \to Y$, where Y is compact but X is not. **(3 marks)**

(e) A space X and points $a, b \in X$ such that $\pi_1(X)$ is nonabelian but the space $Y = X \setminus \{a, b\}$ is simply connected. **(5 marks)**

(f) A continuous bijection that is not a homeomorphism. **(5 marks)**

Solution:

(a) For path connected spaces Y and Z, the product $Y \times Z$ is also path connected and has $H_1(Y \times Z) = H_1(Y) \oplus H_1(Z)$. The spaces S^1 and $\mathbb{R}P^2$ are path connected with $H_1(S^1) = \mathbb{Z}$ [1]and $H_1(\mathbb{R}P^2) = \mathbb{Z}/2$ [2] so $H_1(S^1 \times \mathbb{R}P^2) = \mathbb{Z} \oplus (\mathbb{Z}/2)$ [1].

(b) The simplest example is $X = \mathbb{R}^2$, $a = (-1, 0)$, $b = (0, 0)$, $c = (1, 0)$. It is also easy to exhibit one-dimensional examples (eg the wedge of three circles), and this may well be the most popular type of answer. [3] **[similar examples seen]**

(c) Take $X = S^2$, and let a be any point in X. Then $\pi_1(X) = 0$. Moreover, $X \setminus \{a\}$ is homeomorphic to $\mathbb{R}^2$, which is contractible, so $\pi_1(X \setminus \{a\})$ is again trivial. [5] **The individual facts mentioned are in the summary.**

(d) Take $X = \mathbb{R}$, $Y = \{0\}$, $f(x) = 0$. I expect that students will generally give more complicated examples. [3] **[unseen]**

(e) Let X be the figure eight [2], or in other words the union of the circles of radius one centred at $(1, 0)$ and $(-1, 0)$, so $\pi_1(X)$ is nonabelian [1]. Put $a = (-2, 0)$ and $b = (2, 0)$, so $X \setminus \{a, b\}$ is homeomorphic to the union of two lines meeting at a point. This means that $X \setminus \{a, b\}$ is contractible, and thus simply connected [2]. **The space X is mentioned repeatedly as an example with nonabelian fundamental group, and no other examples are given.**

(f) Define $e\colon [0, 2\pi) \to S^1$ by $e(t) = \exp(i\theta)$ [1]. Every point $z \in S^1$ can be written as $z = \exp(i\theta)$ for a unique angle θ in the range $0 \leq \theta < 2\pi$, so e is a bijection [1]. It is well-known to be continuous [1], but e^{-1} is not continuous [1] because $\exp(-i/n) \to 1$ in S^1 but $e^{-1}(\exp(-i/n)) = 2\pi - 1/n \not\to 0$ [1]. **[bookwork]**

(39) 2020-21 Q1: Give examples as follows, justifying your answers.

(a) Topological spaces X and Y, together with injective functions $f\colon X \to Y$ and $g\colon Y \to X$ such that f, $f \circ g$ and $g \circ f$ are all continuous, but g is not continuous. **(4 marks)**

(b) A compact, path-connected space X together with a continuous map $f\colon X \to X$ with no fixed points. **(4 marks)**

(c) A space X such that $H_1(X)$ is not a free abelian group. (Note here that the zero group is free abelian with no generators, so in particular $H_1(X)$ must be nonzero.) **(4 marks)**

(d) A space X together with points $a, b, c \in X$ such that $|\Pi(X; a, b)| \neq |\Pi(X; b, c)|$. **(4 marks)**

(e) A space X such that $\pi_1(X)$ is a free group with 3 generators, and $H_2(X) = \mathbb{Z}$. **(4 marks)**

Solution: In each case, two marks will be awarded for a correct example, and two further marks for justifying it. Up to two marks may also be awarded for intelligent discussion of an incorrect example. Note that in addition to the main lecture notes, students have access to a two-page summary of examples.

(a) We can use the standard example of a continuous bijection that is not a homeomorphism (Example 4.8):

$$X = (-\infty, 0] \cup (1, \infty) \qquad\qquad Y = \mathbb{R}$$

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x - 1 & \text{if } x > 1 \end{cases} \qquad\qquad g(y) = \begin{cases} y & \text{if } y \leq 0 \\ y + 1 & \text{if } y > 0. \end{cases}$$

Here f is continuous because the domains of the two clauses are both open in X, and $f \circ g$ and $g \circ f$ are identity maps so they are certainly continuous, but g is discontinuous at $y = 0$. [4]

(b) We can take $X = S^n$ for any $n > 0$, and $f(x) = -x$. (Example 9.15 mentions that S^n is compact, as an easy application of Proposition 9.14. It is path-connected by Proposition 5.11. This example of a fixed-point-free endomorphism is mentioned in the solution to Exercise 3 of Problem Sheet 9.) [4]

(c) We can take $X = \mathbb{R}P^2$, then $H_1(X) = \mathbb{Z}/2$, which is not free abelian. (Example 12.15 shows that $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$, and Theorem 18.18 shows that $H_1(\mathbb{R}P^2)$ is the abelianisation of this, which is $\mathbb{Z}/2$ again.) [4]

(d) We can take $X = \{0\} \amalg \mathbb{R}P^2$, with $a = 0$ and $b = c =$ basepoint of $\mathbb{R}P^2$. Then $\Pi(X; a, b) = \emptyset$ and $\Pi(X; b, c) = \pi_1(\mathbb{R}P^2, b) = C_2$ so $|\Pi(X; a, b)| = 0$ but $|\Pi(X; b, c)| = 2$. [4]

(e) We can take $X = S^1 \vee S^1 \vee S^1 \vee S^2$. Using Corollary 15.20 (a special case of the van Kampen Theorem) we see that $\pi_1(X)$ is the free product of three copies of $\pi_1(S^1) = \mathbb{Z}$ together with one copy of $\pi_1(S^2) = 1$, so it is free on three generators. Similarly, we can use Lemma 21.4 (a special case of the Mayer-Vietoris Theorem) to show that $H_2(X) = 0 \oplus 0 \oplus 0 \oplus \mathbb{Z} = \mathbb{Z}$ as required. [4]

Feedback: For part (a), another good answer (given by several students) is to define $f\colon [0, 2\pi) \to S^1$ by $f(x) = e^{ix}$, note that this is bijective, and take $g = f^{-1}$. Most people answered (b) correctly, using the same example as in the solution above. Some people gave answers for (c) where they claimed that $H_1(X)$ was not abelian, but homology groups are always abelian. Most people answered (d) correctly (but sometimes with inadequate justification); correct answers for (e) were rare.

9 Real projective space

In the current version of the course, in the Introduction we define $\mathbb{R}P^n = S^n/(x \sim -x)$ and

$$P_n = \{A \in M_{n+1}(\mathbb{R}) \mid A^2 = A^T = A,\ \text{trace}(A) = 1\},$$

and we mention that $\mathbb{R}P^n$ is homeomorphic to P_n. A proof is given in Problem Sheet 5. In some earlier versions of the course, $\mathbb{R}P^n$ was just defined to be the same as P_n. Problems in this section should be approached from that point of view.

(40)

(a) Define the set $\mathbb{R}P^2$ and the map $q\colon S^2 \to \mathbb{R}P^2$.

(b) Define the usual metric on $\mathbb{R}P^2$, and prove that it is a metric.

(c) Define the space Δ_2, and prove carefully that there is a surjective continuous map $f\colon \mathbb{R}P^2 \to \Delta_2$ satisfying $fq(u, v, w) = (u^2, u^2 + v^2)$ for all $(u, v, w) \in \Delta_2$. You may use general theorems provided that you state them precisely.

Solution:

(a) We can define an equivalence relation $\sim$ on S^2 by $x \sim y$ iff ($x = y$ or $x = -y$). The set $\mathbb{R}P^2$ is the set of equivalence classes for this relation. The map $q\colon S^2 \to \mathbb{R}P^2$ is defined by $q(x) = \langle x\rangle$, the equivalence class of x.

(b) We define $e\colon \mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}$ by $e(x, y) = \min(\|x - y\|, \|x + y\|)$. This is clearly nonnegative and symmetric, and we have $e(x, y) = 0$ iff one of $\|x - y\|$ and $\|x + y\|$ is zero, iff either $x = -y$ or $x = y$, or in other words iff $x \sim y$. Clearly also

$$e(x, y) = e(-x, y) = e(x, -y) = e(-x, -y),$$

and it follows that there is a well-defined function $d\colon \mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}$ such that $d(q(x), q(y)) = e(x, y)$. It is clear that $d(u, v) \geq 0$, with equality iff $u = v$, and that $d(u, v) = d(v, u)$. All that is left is to check the triangle inequality. Suppose we have $u, v, w \in \mathbb{R}P^2$. Choose $x \in S^2$ such that $u = q(x)$. Next, choose $y \in S^2$ such that $q(y) = v$. After replacing y by $-y$ if necessary, we may assume that $\|x - y\| \leq \|x + y\|$, so that $d(u, v) = \|x - y\|$. Next, choose $z \in S^2$ such that $q(z) = w$. After replacing z by $-z$ if necessary, we may assume that $\|y - z\| \leq \|y + z\|$, so that $d(v, w) = \|y - z\|$. We then have

$$\begin{aligned} d(u, w) &= \min(\|x - z\|, \|x + z\|) \\ &\leq \|z - x\| \\ &\leq \|y - x\| + \|z - y\| \\ &= d(u, v) + d(v, w), \end{aligned}$$

as required.

(c) The space Δ_2 is $\{(t_1, t_2) \in \mathbb{R}^2 \mid 0 \leq t_1 \leq t_2 \leq 1\}$. Given a point $a = (u, v, w) \in S^2$ we have $u^2, v^2, w^2 \geq 0$ so $0 \leq u^2 \leq u^2 + v^2 \leq 1 = u^2 + v^2 + w^2$, so $(u^2, u^2 + v^2) \in \Delta_2$. We can thus define a map $g\colon S^2 \to \Delta_2$ by $g(u, v, w) = (u^2, u^2 + v^2)$. The components of g are polynomial functions, so g is continuous. Moreover, $g(-u, -v, -w) = g(u, v, w)$, or in other words $g(-a) = g(a)$, or in other words $g(a) = g(b)$ whenever $a \sim b$. Thus, there is a well-defined function $f\colon \mathbb{R}P^2 \to \Delta_2$ defined by $f(q(a)) = g(a)$. Any function $h\colon \mathbb{R}P^2 \to Y$ is continuous iff $hq\colon S^2 \to Y$ is continuous. We have seen that $g = fq$ is continuous, so f is continuous. Moreover, for any $(t_1, t_2) \in \Delta_2$ we have $(\sqrt{t_1}, \sqrt{t_2 - t_1}, \sqrt{1 - t_2}) \in S^2$ and $fq(\sqrt{t_1}, \sqrt{t_2 - t_1}, \sqrt{1 - t_2}) = (t_1, t_2)$, so f is surjective.

(41)

(a) Define the set $\mathbb{R}P^n$, and write down a metric on it, proving that your formula is well-defined. (You need not show that it is a metric.) **(6 marks)**

(b) Define what it means for a metric space X to be *sequentially compact*. **(3 marks)**

(c) Define the set $\pi_0(X)$, and say what it means for X to be *path-connected*. **(6 marks)**

(d) Prove that the space $\mathbb{R}P^n$ is sequentially compact and path-connected. State clearly any general theorems or results that you use. **(10 marks)**

Solution:

(a) The set $\mathbb{R}P^n$ is the quotient of S^n by the equivalence relation $\sim$, where $x \sim y$ iff ($x = y$ or $x = -y$); we write q for the obvious map $S^n \to \mathbb{R}P^n$. We can define a function $e\colon S^n \times S^n \to [0, \infty)$ by

$$e(x, y) = \min(\|x - y\|_2, \|x + y\|_2) = \min(d_2(x, y), d_2(x, -y)).[2]$$

It is easy to see that

$$e(x, y) = e(-x, y) = e(x, -y) = e(-x, -y).[2]$$

Now suppose we have $a, b \in \mathbb{R}P^n$. We can choose $x, y \in S^n$ such that $q(x) = a$ and $q(y) = b$; these elements are unique up to sign. It follows from the above equation that the value of $e(x, y)$ is independent of the signs, so we may define $d(a, b) = e(x, y)$. This gives a metric on $\mathbb{R}P^n$. [2] [**bookwork**]

(b) A space X is *sequentially compact* if every sequence in X has a convergent subsequence. [3] [**bookwork**]

(c) We define a relation on X by $x \sim y$ iff there is a path in X joining x to y, in other words a continuous map $s\colon I \to X$ with $s(0) = x$ and $s(1) = y$ [2]. Using constant paths we see that this is reflexive, using path reversal we see that it is symmetric, and using path join we see that it is transitive. It is thus an equivalence relation [1], so we can define a quotient set $X/\sim$; this is called $\pi_0(X)$ [1].

We say that X is path-connected if $\pi_0(X)$ is a one-point set, or equivalently if $x \sim y$ for all $x, y \in X$. [2] [**bookwork**]

(d) We have a surjective [1]continuous [1]map $q\colon S^n \to \mathbb{R}P^n$. The set S^n is bounded and closed in $\mathbb{R}^{n+1}$, so it is sequentially compact [2]. A continuous image of a sequentially compact set is sequentially compact [1], so $\mathbb{R}P^n$ is sequentially compact [1]. Also, the space S^n is path-connected (by using great circles, say) [2] and a continuous image of a path-connected set is path-connected [1], so $\mathbb{R}P^n$ is path-connected [1]. [**unseen**]

10 Multipart questions

(42)

(a) What is a *metric space*? What is a *continuous function*?

(b) Define the discrete metric on a set X.

(c) Let X be a space with a discrete metric. Show that any path $s\colon \Delta_1 \to X$ is constant, and deduce that $\pi_0(X) = X$.

(d) Consider the space $Y = \{(x, y) \in \mathbb{R}^2 \mid xy \neq 0\}$ and show that $\pi_0(Y)$ has precisely four elements. If $f\colon Y \to Y$ denotes reflection in the line $x = y$, describe the map $f_*\colon \pi_0(Y) \to \pi_0(Y)$. Is f homotopic to the identity map?

Solution:

(a) A metric space is a set X equipped with a metric, ie a function $d\colon X \times X \to \mathbb{R}$ such that

- $d(x, y) \geq 0$ for all $x, y \in X$, with equality iff $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A function $f\colon X \to Y$ between metric spaces is continuous if for each sequence (x_n) in X that converges to a point $x \in X$, the resulting sequence $(f(x_n))$ in Y converges to the point $f(x)$.

(b) The discrete metric on a set X is defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

(c) Let $s\colon \Delta_1 \to X$ be a path. Define $f\colon \Delta_1 \to \mathbb{R}$ by $f(t) = d(s(t), s(0))$, so f is continuous and $f(0) = 0$. As d can only take the values 0 and 1, we see that f can only take the vales 0 and 1, so by the Intermediate Value Theorem it must be constant. As $f(0) = 0$ we see that $f(t) = 0$ for all t. As $d(s(0), s(t)) = 0$ we see that $s(t) = s(0)$ for all t, in other words s is constant.

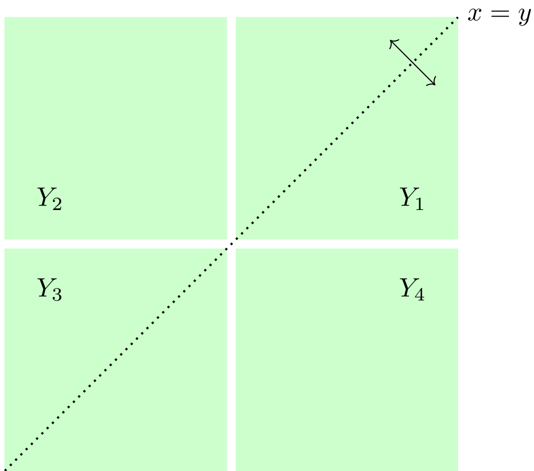
As usual we write $x \sim y$ if x can be connected to y by a path, so $\sim$ is an equivalence relation and $\pi_0(X) = X/\sim$. As the only paths are constant, if $x \sim y$ we must have $x = y$. Thus, each equivalence class consists of just a single point, so $\pi_0(X)$ can be identified with X.

(d) Define

$$\begin{aligned} Y_1 = &\text{ 1st quadrant } = \{(x,y) \mid x > 0, y > 0\} \\ Y_2 = &\text{ 2nd quadrant } = \{(x,y) \mid x < 0, y > 0\} \\ Y_3 = &\text{ 3rd quadrant } = \{(x,y) \mid x < 0, y < 0\} \\ Y_4 = &\text{ 4th quadrant } = \{(x,y) \mid x > 0, y < 0\}. \end{aligned}$$

These are all nonempty convex sets and thus path-connected, and clearly they are open and disjoint. If $(x,y) \in Y$ then $xy \neq 0$ so $x \neq 0$ and $y \neq 0$ so $x < 0$ or $x > 0$ and $y < 0$ or $y > 0$. It follows that (x,y) lies in one of the sets Y_i, so $Y = Y_1 \cup Y_2 \cup Y_3 \cup Y_4$. A path in Y has the form $s(t) = (u(t), v(t))$, where for all t we have $u(t) \neq 0$ and $v(t) \neq 0$. By the intermediate value theorem, we see that $u(0)$ has the same sign as $u(1)$, and $v(0)$ has the same sign as $v(1)$, so if $s(0) \in Y_i$ then $s(1) \in Y_i$ also. It follows that the sets Y_i are the path components of Y, so $\pi_0(Y) = \{Y_1, Y_2, Y_3, Y_4\}$. The formula for the map f is $f(x,y) = (y,x)$, and it follows easily that

$$f_*(Y_1) = Y_1 \qquad f_*(Y_2) = Y_4 \qquad f_*(Y_3) = Y_3 \qquad f_*(Y_4) = Y_2.$$

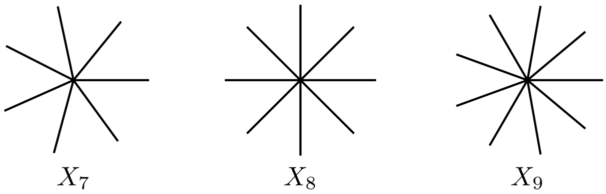


As f_* is not the identity map, we see that f is not homotopic to the identity.

(43) 2018-19 Q1:

(a) Given a topological space X, define the set $\pi_0(X)$. You should include a proof that the relevant equivalence relation is in fact an equivalence relation. **(8 marks)**

(b) Consider $[0,1]$ as a based space with 0 as the basepoint. For $n \geq 3$ we define $X_n = \{z \in \mathbb{C} \mid z^n \in [0,1]\}$:



(i) For which n and m (with $n, m \geq 3$) is X_n homotopy equivalent to X_m? **(3 marks)**

(ii) For which n and m (with $n, m \geq 3$) is X_n homeomorphic to X_m? **(4 marks)**

Justify your answers carefully.

(c) Give examples as follows, with justification:

(1) A based space W with $|\pi_1(W)| = 8$. **(3 marks)**

(2) A space X with two points $a, b \in X$ such that $\pi_1(X, a)$ is not isomorphic to $\pi_1(X, b)$. **(3 marks)**

(3) A space Y such that $H_0(Y) \simeq H_2(Y) \simeq H_4(Y) \simeq H_6(Y) \simeq \mathbb{Z}$ and all other homology groups are trivial. **(4 marks)**

Solution:

(a) We define a relation on X by declaring that $x \sim y$ if there is a continuous path $u\colon [0,1] \to X$ with $u(0) = x$ and $u(1) = y$. [1]

- For any $x \in X$ we can define $c\colon [0,1] \to X$ by $c(t) = x$ for all t. Using this we see that $x \sim x$, so or relation is reflexive. [1]
- Suppose that $x \sim y$, as witnessed by a path u from x to y. The reversed path $\overline{u}(t) = u(1-t)$ is also continuous, with $\overline{u}(0) = y$ and $\overline{u}(1) = x$, which shows that $y \sim x$. This shows that our relation is symmetric. [2]
- Suppose that $x \sim y$ and $y \sim z$, as witnessed by a path u from x to y and a path v from y to z. We can define the concatenated path $u * v\colon [0,1] \to X$ by $(u*v)(t) = u(2t)$ for $0 \leq t \leq 1/2$ and $(u*v)(t) = v(2t-1)$ for $1/2 \leq t \leq 1$ [2] (so in particular $(u*v)(1/2) = y = u(1) = v(0)$). This is continuous on the closed sets $[0, 1/2]$ and $[1/2, 1]$, which cover $[0,1]$, so it is continuous on $[0,1]$. As $(u*v)(0) = u(0) = x$ and $(u*v)(1) = v(1) = z$ we see that $x \sim z$. This shows that our relation is transitive. [1]

We now see that we have an equivalence relation, so we can define $\pi_0(X) = X/\sim$. [1]**[All bookwork]**

(b) (i) For any n we have a contraction of X_n to 0 given by $h(t, z) = tz$ for $0 \leq t \leq 1$. Thus, all the spaces X_n are homotopy equivalent to a point and thus to each other. [3] **[Unseen but easy]**

(ii) Note that $|\pi_0(X_n \setminus \{a\})|$ is 2 for most values of a, but it is n if $a = 0$, and 1 if $|a| = 1$. If we have a homeomorphism $f\colon X_n \to X_m$ then we get a homeomorphism $X_n \setminus \{0\} \to X_m \setminus \{f(0)\}$ so

$$n = |\pi_0(X_n \setminus \{0\})| = |\pi_0(X_m \setminus \{f(0)\})| \in \{1, 2, m\}.$$

As $n, m \geq 3$ this can only occur if $n = m$. Thus, no two of the spaces X_n are homeomorphic. [4] **[Unseen, but the general technique has been seen.]**

(c) (1) We can take $W = (\mathbb{R}P^2)^3$ [2], so $\pi_1(W) = \pi_1(\mathbb{R}P^2)^3 = (\mathbb{Z}/2)^3$, so $|\pi_1(W)| = 8$. [1]**[Unseen, but $\mathbb{R}P^2$ is a standard example.]**

(2) We can take $X = S^1 \cup \{0\} \subset \mathbb{C}$ and $a = 0$ and $b = 1$, so $\pi_1(X, a) = 0$ and $\pi_1(X, b) = \mathbb{Z}$. [3] **[Unseen]**

(3) We can take $Y = S^2 \vee S^4 \vee S^6$. This is connected, so $H_0(Y) = \mathbb{Z}$. For $i > 0$ we have $H_i(Y) = H_i(S^2) \oplus H_i(S^4) \oplus H_i(S^6)$. We also have $H_i(S^i) = \mathbb{Z}$, and $H_i(S^j) = 0$ for $j \neq i$; it follows that $H_*(Y)$ is as required. [4] Alternatively, we can take $Y = \mathbb{C}P^3$. **[Similar examples have been seen.]**

(44) 2018-19 Q3: Let K and L be abstract simplicial complexes.

(a) Define what is meant by a *simplicial map* from K to L. **(3 marks)**

(b) Let $s, t\colon K \to L$ be simplicial maps. Define what it means for s and t to be *directly contiguous*. **(3 marks)**

(c) Prove that if s and t are directly contiguous, then the resulting maps $|s|, |t|\colon |K| \to |L|$ are homotopic. **(3 marks)**

(d) Prove that if s and t are directly contiguous, then the resulting maps $s_*, t_*\colon H_*(K) \to H_*(L)$ are the same. (You can prove the main formula just for $n = 3$ rather than general n.) **(9 marks)**

(e) How many injective simplicial maps are there from $\partial\Delta^2$ to itself? Show that no two of them are directly contiguous. **(7 marks)**

Solution:

(a) A simplicial map from K to L is a function $s\colon \mathrm{vert}(K) \to \mathrm{vert}(L)$ such that whenever $\sigma = \{v_0, \dots, v_n\}$ is a simplex of K, the image $s(\sigma) = \{\sigma(v_0), \dots, \sigma(v_n)\}$ is a simplex of L. [3]

(b) We say that s and t are directly contiguous if whenever $\sigma = \{v_0, \dots, v_n\}$ is a simplex of K, the set

$$s(\sigma) \cup t(\sigma) = \{s(v_0), \dots, s(v_n), t(v_0), \dots, t(v_n)\}$$

is a simplex of L. [3] **[Bookwork]**

(c) Suppose that s and t are directly contiguous. Consider a point $x \in |K|$, so $x \in |\sigma|$ for some $\sigma \in \mathrm{simp}(K)$. Put $\tau = s(\sigma) \cup t(\sigma)$, which is a simplex of L because of the contiguity condition. Both $|s|(x)$ and $|t|(x)$ lie in $|\tau|$, so the whole line segment from $|s|(x)$ to $|t|(x)$ lies in $|\tau|$. We can therefore define a linear homotopy $h\colon [0,1] \times |K| \to |L|$ from $|s|$ to $|t|$ by $h(r, x) = (1-r)|s|(x) + r\,|t|(x)$. [3] **[Bookwork]**

(d) Suppose again that s and t are directly contiguous. Define $u\colon C_nK \to C_{n+1}L$ by

$$u\langle v_0, \dots, v_n\rangle = \sum_{i=0}^{n} (-1)^i \langle s(v_0), \dots, s(v_i), t(v_i), \dots, t(v_n)\rangle. \textbf{[2]}$$

We claim that $du + ud = t_\# - s_\#$ [1]. We will prove this for a generator $x = \langle v_0, v_1, v_2, v_3\rangle \in C_3(K)$, using the abbreviated notation i for v_i or $s(v_i)$, and $\bar{i}$ for $t(v_i)$. We have

$$u(x) = \; +0\overline{0123} \quad -01\overline{123} \quad +012\overline{23} \quad -0123\overline{3} \qquad\qquad d(x) = \; +123 \; -023 \; +013 \; -012$$

$$du(x) = \begin{array}{cccc}
+\overline{0123} & -1\overline{123} & +12\overline{23} & -123\overline{3} \\
-0\overline{123} & +0\overline{123} & -02\overline{23} & +023\overline{3} \\
+0\overline{023} & -01\overline{23} & +01\overline{23} & -013\overline{3} \\
-0\overline{013} & +01\overline{13} & -012\overline{3} & +012\overline{3} \\
+0\overline{012} & -01\overline{12} & +012\overline{2} & -0123
\end{array}
\qquad
ud(x) = \begin{array}{ccc}
+1\overline{123} & -12\overline{23} & +123\overline{3} \\
-0\overline{023} & +02\overline{23} & -023\overline{3} \\
+0\overline{013} & -01\overline{13} & +013\overline{3} \\
-0\overline{012} & +01\overline{12} & -012\overline{2}
\end{array}$$

Most terms cancel in the indicated groups, leaving $du(x) + ud(x) = \overline{0123} - 0123$. In the original notation, this says that

$$(du + ud)(x) = \langle t(v_0), t(v_1), t(v_2), t(v_3)\rangle - \langle s(v_0), s(v_1), s(v_2), s(v_3)\rangle = t_\#(x) - s_\#(x),$$

which means that u is a chain homotopy between $s_\#$ and $t_\#$ [5]. As these maps are chain-homotopic, they induce the same homomorphism between homology groups. [1]**[Bookwork]**

(f) The injective simplicial maps from $\partial\Delta^2$ to itself are just given by permuting the three vertices, so there are $3! = 6$ such maps [2]. Suppose that f and g are permutations that are contiguous. Then the set $f(\{0,1\}) \cup g(\{0,1\})$ must be a simplex, so it has size at most two. However, $f(\{0,1\})$ and $g(\{0,1\})$ both have size two already, so this is only possible if $f(\{0,1\}) = g(\{0,1\})$. As f and g are permutations, it follows that $f(2) = g(2)$. By applying the same logic to $\{0,2\}$ and then $\{1,2\}$, we also see that $f(1) = g(1)$ and $f(0) = g(0)$. Thus, we actually have $f = g$ [5]. **[Unseen]**

(45) 2018-19 Q5: Consider a simplicial complex K with subcomplexes L and M such that $K = L \cup M$. Use the following notation for the inclusion maps:

$$\begin{tikzcd}
L \cap M \arrow[r, "i"] \arrow[d, "j"'] & L \arrow[d, "f"] \\
M \arrow[r, "g"'] & K.
\end{tikzcd}$$

(a) State the Seifert-van Kampen Theorem (in a form applicable to simplicial complexes and subcomplexes as above). **(4 marks)**

(b) State the Mayer-Vietoris Theorem. **(5 marks)**

(c) State a theorem about the relationship between π_1 and H_1. **(3 marks)**

(d) Suppose that $|L|$, $|M|$ and $|L \cap M|$ are all homotopy equivalent to S^1. Suppose that the maps i and j both have degree two.

  (1) Find a presentation for $\pi_1(|K|)$. **(3 marks)**

  (2) Find $H_*(K)$. In particular, you should express each nonzero group as a direct sum of terms like $\mathbb{Z}$ or $\mathbb{Z}/n$. **(10 marks)**

Solution:

(a) Suppose that $|L \cap M|$ is connected and that we have presentations

$$\begin{aligned} \pi_1(|L|) &= \langle x_1, \dots, x_p \mid u_1 = \dots = u_k = 1 \rangle \\ \pi_1(|M|) &= \langle y_1, \dots, y_q \mid v_1 = \dots = v_l = 1 \rangle \\ \pi_1(|L \cap M|) &= \langle z_1, \dots, z_r \mid w_1 = \dots = w_m = 1 \rangle. \end{aligned}$$

Then we have a presentation of $\pi_1(|K|)$ with generators $x_1, \dots, x_p, y_1, \dots, y_q$ and relations $u_1 = \dots = u_r = v_1 = \dots = v_l = 1$ and $i_*(z_t) = j_*(z_t)$ for all t. [4] **[Bookwork]**

(b) There is a natural map $\delta \colon H_n(K) = H_n(L \cup M) \to H_{n-1}(L \cap M)$ such that the resulting sequence

$$H_{n+1}(L \cup M) \xrightarrow{\delta} H_n(L \cap M) \xrightarrow{\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]} H_n(L) \oplus H_n(M) \xrightarrow{[\,f_* \ g_*\,]} H_n(L \cup M) \xrightarrow{\delta} H_{n-1}(L \cap M)$$

is exact for all n [5]. **[Bookwork]**

(c) If $|K|$ is connected [1], then $H_1(K)$ is naturally isomorphic to the abelianisation of $\pi_1(|K|)$ [2]. **[Bookwork]**

(d) (1) As $|L \cap M| \simeq S^1$, we can choose a generator z for $\pi_1(|L \cap M|)$. As i has degree two we see that there is a generator x of $\pi_1(|L|)$ with $i_*(z) = x^2$. As j has degree two we see that there is a generator y of $\pi_1(|M|)$ with $j_*(z) = y^2$. The Seifert-van Kampen Theorem now gives $\pi_1(|K|) = \langle x, y \mid x^2 = y^2 \rangle$. [3] **[Similar examples have been seen.]**

  (2) We have a Mayer-Vietoris sequence as follows:

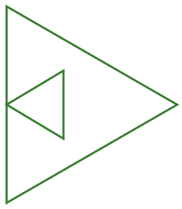
$$\begin{tikzcd} H_2(L \cap M) \arrow[r, "\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]"] & H_2(L) \oplus H_2(M) \arrow[r, "{[\,f_* \ g_*\,]}"] & H_2(K) \arrow[dll] \\ H_1(L \cap M) \arrow[r, "\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]"] & H_1(L) \oplus H_1(M) \arrow[r, "{[\,f_* \ g_*\,]}"] & H_1(K) \arrow[dll] \\ H_0(L \cap M) \arrow[r, "\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]"] & H_0(L) \oplus H_0(M) \arrow[r, two heads, "{[\,f_* \ g_*\,]}"] & H_0(K). \end{tikzcd}$$ [3]

  The spaces $|L \cap M|$, $|L|$ and $|M|$ are all homotopy equivalent to S^1 and so have $H_0 = H_1 = \mathbb{Z}$ and all other homology groups are zero. We also know that i_* and j_* act as the identity on H_0, and as multiplication by 2 on H_1. The sequence therefore has the following form:

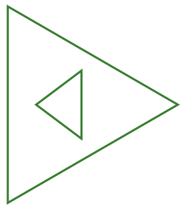
$$\begin{tikzcd} 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & H_2(K) \arrow[dll] \\ \mathbb{Z} \arrow[r, "\left[\begin{smallmatrix} 2 \\ -2 \end{smallmatrix}\right]"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "{[\,f_* \ g_*\,]}"] & H_1(K) \arrow[dll] \\ \mathbb{Z} \arrow[r, "\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, two heads, "{[\,f_* \ g_*\,]}"] & H_0(K). \end{tikzcd}$$ [3]

From this we can read off that $H_2(K) = 0$ and $H_0(K) = \mathbb{Z}$ [1]and that $H_1(K) = \mathbb{Z}^2/\mathbb{Z}.(2, -2)$ [1]. If we use the basis $\{(1, 0), (1, -1)\}$ for $\mathbb{Z}^2$ we get $H_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z}/2$ [1]. By extending the sequence further upwards, it is also clear that $H_n(K) = 0$ for $n > 2$ [1]. **[Similar examples have been seen.]**

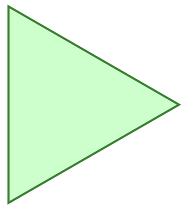
(46) 2019-20 Q1: Consider the following spaces:



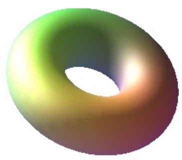
X_0



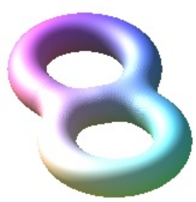
X_1



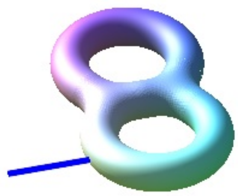
X_2



X_3



X_4



X_5

$$X_6 = (S^1 \times S^1) \setminus \{(1, 1)\} \qquad X_7 = GL_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \mid \det(A) \neq 0\}$$
$$X_8 = \mathbb{R} \qquad X_9 = \{(u, v) \in \mathbb{C}^2 \mid 1 \leq |u| \leq 2 \leq |v| \leq 3\}.$$

(Here X_3 and X_4 are closed orientable surfaces, and X_5 is the union of X_4 with a line segment with one endpoint lying on X_4. Everything else should be clear.)

(a) These 10 spaces can be grouped into 5 pairs $\{X_i, X_j\}$ such that X_i is homotopy equivalent to X_j. Find these pairs, and justify your answers. In each case you should prove that X_i is homotopy equivalent to X_j, and also that it is not homotopy equivalent to any of the other spaces. **(25 marks)**

(b) For each pair $\{X_i, X_j\}$ as in (a), prove that X_i is not homeomorphic to X_j. (In one case you may need to appeal to some geometric intuition, but you should be able to give a more formal proof in the other four cases.) **(15 marks)**

Solution:

(a) **This will need to be marked as a whole. There will be 5 marks for correct identification of the pairs, 10 marks for justifying why they are homotopy equivalent, and a further 10 marks for explaining why there are no further equivalences.** [15]

- X_0 consists of two circles meeting at a single point and so is homeomorphic to the figure eight. This is in turn homotopy equivalent to the punctured torus X_6, as explained in Example 15.26 and the associated interactive demonstration.
- X_1 is homeomorphic to the union of two disjoint circles. On the other hand, Example 4.9 shows that the space $X_7 = GL_2(\mathbb{R})$ is homeomorphic to $\mathbb{R}^3 \times S^1 \times \{1, -1\}$, so it is homotopy equivalent to $S^1 \times \{1, -1\}$, which is again a union of two disjoint circles. Thus, X_1 is homotopy equivalent to X_7.
- X_2 and X_8 are both contractible and so are homotopy equivalent to each other.
- X_3 is just the torus $S^1 \times S^1$. There is a homeomorphism

$$p\colon [0, 1]^2 \times X_3 = [0, 1]^2 \times S^1 \times S^1 \to X_9$$

given by $p(s, t, u, v) = ((1 + s)u,\ (2 + t)v)$, and $[0, 1]^2$ is contractible, so X_3 is homotopy equivalent to X_9.

– The spaces X_4 and X_5 are homotopy equivalent. Indeed, the extra interval in X_5 can be parametrised as $\{u(t) \mid 0 \leq t \leq 1\}$, with $u(0)$ being the end lying in X_4. We have an evident inclusion $i\colon X_4 \to X_5$ and a retraction $r\colon X_5 \to X_4$ given by $r(u(t)) = u(0)$ and $r(x) = x$ for all $x \in X_4$. Then $r \circ i$ is equal to the identity. We can also define $h\colon [0,1] \times X_5 \to X_5$ by $h(s, u(t)) = u(st)$ and $h(s,x) = x$ for all $x \in X_4$. This gives a homotopy $i \circ r \simeq \mathrm{id}$, so we have a homotopy equivalence as claimed.

If two spaces are homotopy equivalent, then they have isomorphic homology. We can tabulate the homology groups of the X_i as follows:

	H_0	H_1	H_2
X_0, X_6	$\mathbb{Z}$	$\mathbb{Z}^2$	0
X_1, X_7	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
X_2, X_8	$\mathbb{Z}$	0	0
X_3, X_9	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$
X_4, X_5	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}$

As all the lines are different, there are no additional homotopy equivalences [10]. **There are also valid approaches using π_0 and π_1. They are less clear and efficient, but can also be given full marks if done correctly.**

(b) The space X_0 is compact but X_6 is not, so X_0 is not homeomorphic to X_6 [3]. Similarly X_1 is compact but X_7 is not [3], and X_2 is compact but X_8 is not [3]. Next, X_5 can be disconnected by removing a single point, but X_4 cannot, so X_4 and X_5 are not homeomorphic [3]. Finally X_3 and X_9 are not homeomorphic because X_3 is 2-dimensional and X_9 is 4-dimensional [3]. (This is not quite a complete proof, because we have not given a formal definition of dimensionality. The Invariance of Domain Theorem does most of what we need, but a bit more discussion would be required.)

(47) 2019-20 Q2:

(a) Let A and B be finite abelian groups such that $|A|$ and $|B|$ are coprime.

(i) What can you say about homomorphisms from A to B? **(10 marks)**

(ii) Now suppose we have a short exact sequence $A \to U \to B$ of abelian groups. By considering the classification of finite abelian groups, or otherwise, what can you say about U? **(15 marks)**

(b) Let X be a topological space, with open subspaces U and V such that $X = U \cup V$. Suppose that U, V, X and $U \cap V$ are all path-connected, and that for all $k > 0$ we have $H_k(U \cap V) = \mathbb{Z}/2^k$ and $H_k(U) = \mathbb{Z}/3^k$ and $H_k(V) = \mathbb{Z}/5^k$. Calculate $H_*(X)$. **(15 marks)**

Solution:

(a) (i) The only homomorphism from A to B is the zero homomorphism [3]. Indeed, if $\phi\colon A \to B$ is a homomorphism then $\phi(A)$ is a subgroup of B and so has order dividing $|B|$. On the other hand, the First Isomorphism Theorem says that $|\phi(A)| = |A|/|\ker(\phi)|$, and this is a divisor of $|A|$. As $|A|$ and $|B|$ are coprime, we conclude that $|\phi(A)| = 1$, so $\phi(A) = \{0\}$, so $\phi = 0$. [7]

(ii) If $A \xrightarrow{f} U \xrightarrow{g} B$ is a short exact sequence, we claim that $U \simeq A \oplus B$ [3]. Indeed, we have $|U| = |A|.|B|$. We can write U as a direct sum of groups of the form $\mathbb{Z}/p^k$. As $|U| = |A|.|B|$ with $|A|$ and $|B|$ coprime, we see that p must divide $|A|$ or $|B|$ but not both. Let A' be the sum of all the factors where p divides $|A|$, and let B' be the sum of all the factors where p divides $|B|$, so $U = A' \oplus B'$. The homomorphism $f\colon A \to A' \oplus B'$ can be decomposed into a pair of homomorphisms $f_0\colon A \to A'$ and $f_1\colon A \to B'$. The homomorphism $g\colon A' \oplus B' \to B$ can be decomposed into a pair of homomorphisms $g_0\colon A' \to B$ and $g_1\colon B' \to B$. Here f_1 and g_0 are zero by part (i). As $f_1 = 0$ we have $\mathrm{img}(f) \leq A'$, and as $g_0 = 0$ we have $\ker(g) \geq A'$. As the sequence is exact we have $\mathrm{img}(f) = \ker(g)$, so this group must be equal to A'. Also, as f is injective we see that f_0 is injective, and as g is surjective we see that g_1 is surjective. It now follows that f_0 and g_1 are isomorphisms, and thus that $U = A' \oplus B; \simeq A \oplus B$ as claimed. [12]

(b) The connectivity assumptions mean that $H_0(X) = \mathbb{Z}$ and that we have a truncated Mayer-Vietoris sequence [2]. For $k > 1$ this takes the form

$$\mathbb{Z}/2^k \xrightarrow{e} \mathbb{Z}/3^k \oplus \mathbb{Z}/5^k \xrightarrow{f} H_k(X) \xrightarrow{g} \mathbb{Z}/2^{k-1} \xrightarrow{e} \mathbb{Z}/3^{k-1} \oplus \mathbb{Z}/5^{k-1}.\text{[3]}$$

The maps marked e are zero by (a)(i) [2], so f is injective and g is surjective by exactness [4], which means that the middle three terms form a short exact sequence. Thus, (a)(ii) tells us that

$$H_k(X) = \mathbb{Z}/3^k \oplus \mathbb{Z}/5^k \oplus \mathbb{Z}/2^{k-1} = \mathbb{Z}/(30^k/2)[4]$$

(where we have used the Chinese Remainder Theorem to tidy up the final answer a little). This formula remains valid for $k = 1$, although the argument is a tiny bit different.

(48) 2020-21 Q2: Fix $n \geq 2$. Define an equivalence relation on the disc $B^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ by $z_0 \sim z_1$ iff $z_0 = z_1$, or ($|z_0| = |z_1| = 1$ and $z_0^n = z_1^n$). Put $X = B^2/\sim$ and

$$Y = \{(u, v) \in \mathbb{C}^2 \mid |u| \leq 1, \quad v^n = (1 - |u|)^n u\}.$$

Note that when $n = 2$ we just have $X = \mathbb{R}P^2$; this should guide your thinking about the general case.

(a) Show carefully that there is a homeomorphism $f: X \to Y$ such that $f([z]) = (z^n, (1 - |z|^n)z)$ for all $z \in B^2$. You should prove in particular that f is well-defined, injective and surjective, and that both f and f^{-1} are continuous. You may assume that polynomials and the absolute value function are continuous, but beyond that you should not assume any properties of the given formula without proof. **(13 marks)**

(b) For the boundary $S^1 \subset B^2$, explain briefly why $S^1/\sim$ is homeomorphic to S^1 again. **(3 marks)**

(c) By adapting the method used for $\mathbb{R}P^2$, calculate $H_*(X)$. **(14 marks)**

Solution:

(a) Suppose that $z \in B^2$ (so $|z| \leq 1$) and put $u = z^n$ and $v = (1 - |z|^n)z = (1 - |u|)z$. We then have $|u| = |z|^n \leq 1$ and $v^n = (1 - |u|)^n z^n = (1 - |u|)^n u$, so $(u, v) \in Y$ [1]. We can thus define a continuous map $f_0: B^2 \to Y$ by $f_0(z) = (z^n, (1 - |z|^n)z)$. Now suppose we have $z_0, z_1 \in B^2$ with $z_0 \sim z_1$; we claim that $f(z_0) = f(z_1)$ [1]. If $z_0 = z_1$ then this is clear. Otherwise, we must have $|z_0| = |z_1| = 1$ (which means that $f_0(z_i) = (z_i^n, 0)$) and $z_0^n = z_1^n$, so $f_0(z_0) = f_0(z_1)$ as required [1]. By the universal property of quotients (Corollary 8.20) there is a unique continuous map $f: X \to Y$ such that $f([z]) = f_0(z)$ for all z [1].

Now suppose we have $(u, v) \in Y$, so $v^n = (1-|u|)^n u$. If $|u| \neq 1$ then $0 < 1-|u| \leq 1$ and we put $z = v/(1-|u|) \in \mathbb{C}$. The relation $v^n = (1 - |u|)^n u$ becomes $z^n = u$. It follows that $|z|^n = |u| < 1$ so $|z| < 1$ so $z \in B^2$, and we find that $f([z]) = f_0(z) = u$. On the other hand, if $|u| = 1$ then the relation $v^n = (1 - |u|)^n u$ gives $v = 0$. We can let z be any one of the n'th roots of u and we get $|z| = 1$ and $f([z]) = f_0(z) = (u, 0)$. This shows that f is surjective. [3]

Now suppose we have $z_0, z_1 \in B^2$ with $f([z_0]) = f([z_1])$, or in other words $z_0^n = z_1^n$ and $(1 - |z_0|^n)z_0 = (1 - |z_1|^n)z_1$. Put $r = |z_0| \in [0, 1]$. Using $z_0^n = z_1^n$ we get $r^n = |z_1|^n$ so $|z_1|$ is also equal to r. Thus, the equation $(1 - |z_0|^n)z_0 = (1 - |z_1|^n)z_1$ becomes $(1 - r^n)(z_0 - z_1) = 0$. If $r < 1$ this gives $z_0 = z_1$, so certainly $[z_0] = [z_1]$. On the other hand, if $r = 1$ then the relation $z_0^n = z_1^n$ gives $z_0 \sim z_1$ (from the definition of the equivalence relation) and so $[z_0] = [z_1]$. Either way, we have $[z_0] = [z_1]$, so we conclude that f is injective. [3]

Note also that X is a quotient of the compact space B^2, so it is again compact. Moreover, Y is a metric space and so is Hausdorff. As f is a continuous bijection from a compact space to a Hausdorff space, it is a homeomorphism by Proposition 9.28. [3]

(b) For $z \in S^1$ we have $(1-|z|^n)z = 0$, so f restricts to give a homeomorphism $S^1/\sim \to S^1 \times \{0\} \simeq S^1$. Alternatively, on S^1 the equivalence relation is just $z_0 \sim z_1 \iff z_0^n = z_1^n$, so the map $[z] \mapsto z^n$ gives the required homeomorphism. [3]

(c) Put $\widetilde{U} = B^2 \setminus \{0\}$ and $\widetilde{V} = B^2 \setminus S^1 = OB^2$. Let U and V be the images of $\widetilde{U}$ and $\widetilde{V}$ in X. These are open sets which cover X, so they give a Mayer-Vietoris sequence. [3]

The equivalence relation does not do anything to $\widetilde{V}$, so V is just an open disc, which is contractible. Thus, the only nontrivial homology group is $H_0(V) = \mathbb{Z}$ [2]. Next, we can deform $\widetilde{U}$ radially outward onto S^1, and this is compatible with the equivalence relation, so U is homotopy equivalent to $S^1/\sim$, which is homeomorphic to S^1 by (b). Thus, we have $H_0(U) = H_1(U) = \mathbb{Z}$ and all other homology groups are zero [2]. Also, $U \cap V$ is an annulus so $H_0(U \cap V) = H_1(U \cap V) = \mathbb{Z}$ and again all other homology groups are zero [1]. As U, V and $U \cap V$ are connected we can use the truncated version of the Mayer-Vietoris sequence:

$$H_2(U) \oplus H_2(V) \to H_2(X) \to H_1(U \cap V) \to H_1(U) \oplus H_1(V) \to H_1(X) \to H_1(U \cap V) \to 0.[2]$$

Using the above determination of the homology groups, this becomes

$$0 \to H_2(X) \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \to H_1(X) \to 0.[1]$$

The standard circle in the annulus $U \cap V$ gets wrapped n times around the boundary circle $S^1/\sim$, so i_* is multiplication by n, which is injective [1]. It follows that $H_2(X) = 0$ and $H_1(X) = \mathbb{Z}/n$. As X is connected, we have $H_0(X) = \mathbb{Z}$ [1]. For $k > 2$ is is clear from the Mayer-Vietoris sequence that $H_k(X) = 0$. [1]

Feedback:

(a) Very few people checked that $(z^n, (1-|z|^n)z) \in Y$, despite my ranting about this sort of thing in connection with Problem Sheet 10. Very few people distinguished clearly between f_0 and f; in particular, many people claimed to be proving that f is continuous, but actually proved that f_0 is continuous. Attempts to prove that f is well-defined and injective were of variable quality. For surjectivity, many people claimed that $f([u^{1/n}]) = (u, v)$ for all $(u, v) \in Y$. Here everything is complex so we usually have n different choices of z with $z^n = u$, i.e. n different possible values of $u^{1/n}$. If you choose the right one then you will get $f([z]) = (u, v)$, but if you choose the wrong one then you will instead get $f([z]) = (u, e^{2\pi ik/n}v)$ for some $k \neq 0$. Thus, a more detailed argument needs to be given. These issues also mean that f^{-1} is not given by a simple and well-defined formula, so the only reasonable way to prove that f^{-1} is continuous is to use Proposition 9.28. This is all similar to Examples 8.24, 8.26, 9.29 and 9.30 in the notes.

(b) Most people gave answers that were along the right lines.

(c) Most people who made a serious attempt at this got it roughly right; but some people gave up.