The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2022–2023**

Algebraic Topology **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 (a) Explain the terms *homeomorphism* and *homeomorphic*. ***(3 marks)***

(b) Explain the terms *homotopy*, *homotopic* and *homotopy equivalent*, distinguishing carefully between them. ***(5 marks)***

(c) Consider the following spaces:

$$X_0 = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \notin \mathbb{Z}\}$$
$$X_1 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \in \mathbb{Z}\}$$
$$X_2 = \{z \in \mathbb{C} \mid z \notin \mathbb{Z}\}$$
$$X_3 = \{z \in \mathbb{R} \mid z \notin \mathbb{Z}\}$$
$$X_4 = \{z \in \mathbb{C} \mid |z| \in \mathbb{Z}\}.$$

(i) Sketch all these spaces. ***(5 marks)***

(ii) For which pairs (i, j) is X_i homotopy equivalent to X_j? Justify your answer briefly. In cases where X_i is homotopy equivalent to X_j you should explain why, and in cases where X_i is not homotopy equivalent to X_j, you should explain that as well. ***(6 marks)***

(iii) For which pairs (i, j) is X_i homeomorphic to X_j? Justify your answer briefly. In cases where X_i is homeomorphic to X_j you should explain why, and in cases where X_i is not homeomorphic to X_j, you should explain that as well. ***(6 marks)***

2 (a) What does it mean to say that a topological space X is *compact*? If your explanation relies on any auxiliary terms, then you should define them. ***(3 marks)***

(b) Let X be compact topological space, and let Y be a closed subset of X.

(i) Define the subspace topology on Y. ***(2 marks)***

(ii) Prove that when equipped with the subspace topology, Y is again compact. ***(5 marks)***

(iii) Give an example of a compact space X and a compact subpace Y such that Y is not closed in X. ***(3 marks)***

(iv) Explain a commonly-satisfied condition on X that guarantees that compact subspaces are closed. If your explanation relies on any auxiliary terms, then you should define them. However, you need not prove anything. ***(3 marks)***

(c) Put $X = \mathbb{Z} \times \mathbb{Z}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid 100 < x^2 + y^2 < 10000\}$, considered as subspaces of the plane $\mathbb{R}^2$.

(i) Is X compact? ***(1 mark)***

(ii) Is Y compact? ***(1 mark)***

(iii) Is $X \cap Y$ compact? ***(2 marks)***

Justify your answers.

(d) Let X be a metric space such that $X \setminus \{x\}$ is compact for all $x \in X$. Prove that X is finite. ***(5 marks)***

3 Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps.

(a) Define what is meant by saying that the above sequence is short exact. ***(3 marks)***

Now recall that a *snake* for the above sequence is a system (c, w, v, u, a) such that

- $c \in H_n(W)$;
- $w \in Z_n(W)$ is a cycle such that $c = [w]$;
- $v \in V_n$ is an element with $p(v) = w$;
- $u \in Z_{n-1}(U)$ is a cycle with $i(u) = d(v) \in V_{n-1}$;
- $a = [u] \in H_{n-1}(U)$.

(b) Prove that for each $c \in H_n(W)$ there is a snake starting with c. ***(7 marks)***

(c) Explain how the connecting homomorphism $\delta\colon H_n(W) \to H_{n-1}(U)$ is defined in terms of snakes. If any further lemmas are needed to ensure that your definition is meaningful, then you should state those lemmas carefully, but you need not prove them. ***(4 marks)***

(d) Consider the following example. For each $k \in \mathbb{Z}$ we have

$$
\begin{aligned}
U_k &= \mathbb{Z}/24 = \mathbb{Z}/(2^3 \times 3) & \qquad d^U(x) &= 12x = 2^2 \times 3 \times x \\
V_k &= \mathbb{Z}/1296 = \mathbb{Z}/(2^4 \times 3^4) & \qquad d^V(x) &= 36x = 2^2 \times 3^2 \times x \\
W_k &= \mathbb{Z}/54 = \mathbb{Z}/(2 \times 3^3) & \qquad d^W(x) &= -18x = -2 \times 3^2 \times x.
\end{aligned}
$$

The maps

$$U_k \xrightarrow{i} V_k \xrightarrow{p} W_k$$

are $i(a \pmod{24}) = 54a \pmod{1296}$ and $p(b \pmod{1296}) = b \pmod{54}$.

(i) Check that i and p are chain maps. (You may assume that they give a short exact sequence.) ***(3 marks)***

(ii) Calculate the groups $H_k(U)$, $H_k(V)$ and $H_k(W)$. ***(5 marks)***

(iii) By finding an appropriate snake, calculate the homomorphism $\delta\colon H_k(W) \to H_{k-1}(U)$. ***(3 marks)***

4 For each of the following, either give an example (with justification) or prove that no example can exist.

(a) A topological space X with two noncompact subsets Y, Z such that $Y \cup Z$ is compact. ***(5 marks)***

(b) Subsets $A, B, C \subseteq \mathbb{R}^2$ such that A, B, C, $A \cup B$, $A \cup C$ and $B \cup C$ are all contractible, but $A \cup B \cup C$ is not contractible. ***(5 marks)***

(c) A topological space X with two open subsets U and V such that U, V and $U \cap V$ are all homotopy equivalent to S^1, and $X = U \cup V$, and X is homotopy equivalent to S^4. ***(5 marks)***

(d) A path connected space X such that $H_*(X)$ is not isomorphic to $H_*(X \times X)$. ***(5 marks)***

(e) Spaces X and Y such that X is path connected, Y is not path connected, and $H_k(X) \simeq H_k(Y)$ for all k. ***(5 marks)***

5 Consider S^1 as the unit circle in $\mathbb{R}^2$ as usual. Let X be a path connected space, and put

$$U = \{(t,x) \in S^1 \times X \mid t \neq (0,1)\}$$
$$V = \{(t,x) \in S^1 \times X \mid t \neq (0,-1)\}.$$

We use the usual notation for inclusion maps:

$$\begin{tikzcd} U \cap V \arrow[r, "i"] \arrow[d, "j"'] & U \arrow[d, "k"] \\ V \arrow[r, "l"'] & S^1 \times X. \end{tikzcd}$$

(a) Define maps $f, g \colon X \to U \cap V$ such that f gives a homotopy equivalence from X to one path component of $U \cap V$, and g gives a homotopy equivalence from X to the other path component of $U \cap V$. ***(4 marks)***

(b) Prove that the map $i' = i \circ f \colon X \to U$ is homotopic to $i \circ g$, and also that i' is a homotopy equivalence. (You can then assume without further argument that the map $j' = j \circ f \colon X \to V$ is homotopic to $j \circ g$, and that j' is a homotopy equivalence.) ***(6 marks)***

(c) Deduce descriptions (in terms of $H_*(X)$) of the homology groups $H_p(U \cap V)$, $H_p(U)$ and $H_p(V)$, and the homomorphism

$$\alpha = \begin{bmatrix} i_* \\ -j_* \end{bmatrix} : H_p(U \cap V) \to H_p(U) \oplus H_p(V).$$

Find the kernel and image of α. ***(8 marks)***

(d) Show that every element of $H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a), 0) + \alpha(b)$ for a unique pair $(a, b) \in H_p(X) \oplus H_p(X)$. ***(3 marks)***

(e) Deduce that there is a short exact sequence $H_p(X) \to H_p(S^1 \times X) \to H_{p-1}(X)$. ***(4 marks)***

End of Question Paper