Algebraic Topology

(1) For $n \geq 3$, we put $X_n = \mathbb{R}^2 \setminus \{(1,0),(2,0),\dots,(n,0)\}$.

(a) Define the following terms: *topology*, *topological space*, *continuous map*, *homeomorphism*. **(7 marks)**

(b) Find a space Y_n consisting of a finite number of straight line segments that is homotopy equivalent to X_n. Give a brief justification for the claim that Y_n is homotopy equivalent to X_n. **(6 marks)**

(c) Prove that X_n is not homeomorphic to Y_n. **(3 marks)**

(d) Prove that X_n is not homotopy equivalent to S^m for any m. **(4 marks)**

(e) Find contractible open sets $U_n, V_n \subseteq \mathbb{C}$ such that $X_n = U_n \cup V_n$. Give a careful proof that U_n and V_n are contractible. **(5 marks)**

Claims about the homology of particular spaces should be stated clearly and justified briefly, but details are not required.

Solution:

(a) A *topology* on a set X is a family τ of subsets of X (called *open sets*) [1]such that

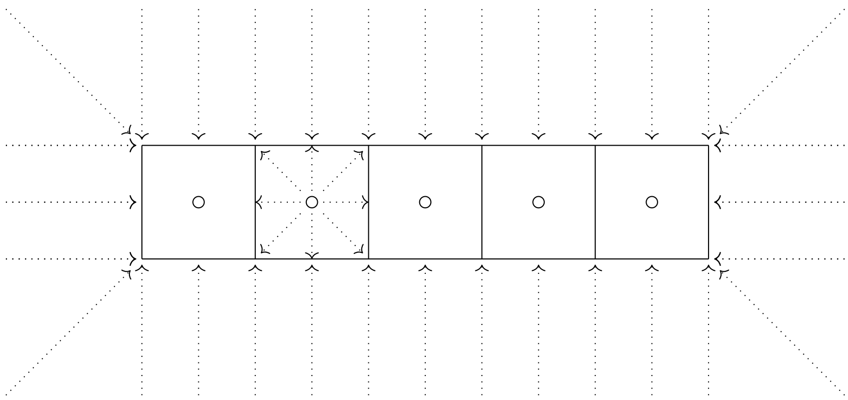
(1) The empty set and the whole set X are open [1]

(2) The union of any family of open sets is open [1]

(3) The intersection of any finite list of open sets is open. [1]

A *topological space* is a st equipped with a topology. If X and Y are topological spaces, a *continuous map* from X to Y is a function $f\colon X \to Y$ such that for every open set $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X [2]. A *homeomorphism* from X to Y is a bijective map $f\colon X \to Y$ with the property that both $f\colon X \to Y$ and $f^{-1}\colon Y \to X$ are continuous [1].

(b) We define Y_n to be the union of line segments from $[\frac{1}{2}, n+\frac{1}{2}] \times \{\pm\frac{1}{2}\}$ and $\{i+\frac{1}{2}\} \times [-\frac{1}{2},\frac{1}{2}]$ for $0 \leq i \leq n$ [3]:



Let $i\colon Y_n \to X_n$ be the inclusion. The dotted arrows indicate a continuous map $r\colon X_n \to Y_n$ such that $ri = \mathrm{id}$ and ir is homotopic to the identity by a straight line homotopy; this proves that Y_n is homotopy equivalent to X_n. [3]

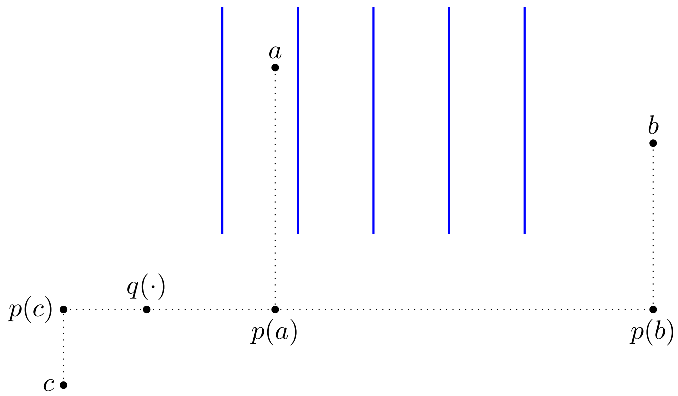
(c) The space Y_n is a bounded and closed subspace of $\mathbb{R}^2$, so it is compact. The space X_n is unbounded and so is not compact. It follows that X_n cannot be homeomorphic to Y_n. [3]

(d) It was proved in the notes that

$$H_i(X_n) \simeq \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}^n & \text{if } i = 1 \\ 0 & \text{otherwise .[2]} \end{cases}$$

In particular, the total rank of all the homology groups of X_n is $n+1 \geq 4$, whereas the total rank of all homology groups of S^m is 2. Homotopy equivalent spaces have isomorphic homology, so X_n cannot be homotopy equivalent to S^m. [2]

(e) Put $A_n = \{1,\dots,n\} \times [0,\infty)$ and $B_n = \{1,\dots,n\} \times (-\infty, 0]$. These are closed subsets of $\mathbb{R}^2$ with $A_n \cap B_n = \{(1,0),\dots,(n,0)\}$. It follows that the sets $U_n = \mathbb{R}^2 \setminus A_n$ and $V_n = \mathbb{R}^2 \setminus B_n$ are open with $U_n \cup V_n = \mathbb{R}^2 \setminus (A_n \cap B_n) = X_n$ [2]. Define $p, q \colon U_n \to U_n$ by $p(x,y) = (x,-1)$ and $q(x,y) = (0,-1)$. If $(x,y) \in U_n$ then the line segment from (x,y) to $p(x,y)$ is vertical, and the line segment from $p(x,y)$ to $q(x,y)$ is horizontal, and neither segment touches A_n. Thus, we have straight line homotopies from the identity to p and then from p to the constant map q, proving that U_n is contractible.



Essentially the same argument (using $r(x,y) = (x,1)$ and $s(x,t) = (0,1)$) proves that V_n is contractible. [3]

(2)

(a) Let X be a topological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is an equivalence relation. **(6 marks)**

(b) Let $f \colon X \to Y$ be a continuous map. Define the induced map $f_* \colon \pi_0(X) \to \pi_0(Y)$, and prove that it is well-defined. **(4 marks)**

(c) Show that if $f, g \colon X \to Y$ are homotopic maps then $f_* = g_* \colon \pi_0(X) \to \pi_0(Y)$. **(4 marks)**

(d) Let Y and Z be topological spaces. Construct a bijection $\pi_0(Y \times Z) \to \pi_0(Y) \times \pi_0(Z)$, and prove that it is a bijection. **(5 marks)**

(e) Define $i \colon \mathbb{Z} \to \mathbb{R} \setminus \mathbb{Z}$ by $i(n) = n + \frac{1}{2}$. Prove that there do not exist continuous maps $\mathbb{Z} \xrightarrow{f} S^2 \times S^2 \xrightarrow{g} \mathbb{R} \setminus \mathbb{Z}$ such that i is homotopic to $g \circ f$. **(6 marks)**

Solution:

(a) We write $x \sim y$ iff there is a path in X from x to y, in other words a continuous map $s \colon I \to X$ such that $s(0) = x$ and $s(1) = y$ [2]. For any $x \in X$ we can define $c_x \colon I \to X$ by $c_x(t) = x$ for all t; this is a path from x to x, proving that $x \sim x$ [1]. If $x \sim y$ then there is a path s from x to y and we can define a path $\overline{s}$ from y to x by $\overline{s}(t) = s(1-t)$; this shows that $y \sim x$ [1]. If there is also a path r from y to z then we can define a path $s * r$ from x to z by

$$(s*r)(t) = \begin{cases} s(2t) & \text{if } 0 \leq t \leq 1/2 \\ r(2t-1) & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

This is well-defined because $s(1) = y = r(0)$, and it is continuous by closed patching [1]. This shows that $x \sim z$ [1]. Thus $\sim$ is reflexive, symmetric and transitive and thus is an equivalence relation.

(b) Let c be an element of $\pi_0(X)$, in other words a path component in X. For any $x \in c$ we have a point $f(x) \in Y$, and thus a path-component $[f(x)] \in \pi_0(Y)$. If x' is another point in c then $x \sim x'$ so we can choose a path s from x to x' in X [1]. Thus $f \circ s \colon I \to Y$ is a path in Y from $f(x)$ to $f(x')$ [1], so $f(x) \sim f(x')$, so $[f(x)] = [f(x')]$ [1]. We can thus define $f_*(c) = [f(x)]$; this is independent of the choice of x and thus is well-defined [1].

(c) If $f, g\colon X \to Y$ are homotopic then we can chooose a map $h\colon I \to X \to Y$ such that $h(0,x) = f(x)$ and $h(1,x) = g(x)$ for all x [1]. If $c \in \pi_0(X)$ we can choose $x \in X$ and note that $f_*(c) = [f(x)]$ and $g_*(c) = [g(x)]$. We can also define a map $s\colon I \to Y$ by $s(t) = h(t,x)$ [2]. This gives a path from $s(0) = f(x)$ to $s(1) = g(x)$, so $[f(x)] = [g(x)]$, in other words $f_*(c) = g_*(c)$ [1].

(d) Suppose we have topological spaces Y and Z. Let $p\colon Y \times Z \to Y$ and $q\colon Y \times Z \to Z$ be the projection maps, defined by $p(y,z) = y$ and $q(y,z) = z$ [1]. Define $\phi\colon \pi_0(Y \times Z) \to \pi_0(Y) \times \pi_0(Z)$ by $\phi(c) = (p_*(c), q_*(c))$, so $\phi([y,z]) = ([y],[z])$ [1]. Any element of $\pi_0(Y) \times \pi(0(Z)$ has the form (b,c), where $b \in \pi_0(Y)$ and $c \in \pi_0(Z)$. We can then choose $y \in Y$ and $z \in Z$ such that $b = [y]$ and $c = [z]$. This gives an element $(y,z) \in Y \times Z$ and a path component $[y,z] \in \pi_0(Y \times Z)$ with $\phi([y,z]) = ([y],[z]) = (b,c)$. This shows that ϕ is surjective [1]. Now suppose we have two path components $[y,z]$ and $[y',z']$ in $\pi_0(Y \times Z)$ which satisfy $\phi([y,z]) = \phi([y',z'])$. This means that $([y],[z]) = ([y'],[z'])$, so $[y] = [y']$ and $[z] = [z']$. As $[y] = [y']$ in $\pi_0(Y)$ we can choose a continuous map $v\colon [0,1] \to Y$ with $v(0) = y$ and $v(1) = y'$. Similarly, we can choose a continuous map $w\colon [0,1] \to Z$ with $w(0) = z$ and $w(1) = z'$. Now define $u\colon [0,1] \to Y \times Z$ by $u(t) = (v(t), w(t))$, noting that this is continuous by the universal property of the product topology. We have $u(0) = (y,z)$ and $u(1) = (y',z')$ so $[y,z] = [y',z']$ in $\pi_0(Y \times Z)$[2]. This proves that ϕ is also injective, and so is a bijection.

(e) Suppose (for a contradiction) that i is homotopic to $g \circ f$ for some continuous maps $\mathbb{Z} \xrightarrow{f} S^2 \times S^2 \xrightarrow{g} \mathbb{R} \setminus \mathbb{Z}$ [1]. It then follows from (c) that $i_* = g_* \circ f_*$ [1]. However, it is standard that S^2 is path connected [1], or equivalently that $|\pi_0(S^2)| = 1$. It follows using (d) that $S^2 \times S^2$ is also path connected [1], so $f_*([-1]) = f_*([0])$ in $\pi_0(S^2 \times S^2)$, so $g_*(f_*([-1])) = g_*(f_*([0]))$, so $i_*([-1]) = i_*([0])$ in $\pi_0(\mathbb{R} \setminus \mathbb{Z})$ [1]. This means that there is a path from $-\frac{1}{2}$ to $\frac{1}{2}$ in $\mathbb{R} \setminus \mathbb{Z}$, which violates the Intermediate Value Theorem[1].

(3)

(a) Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps. Define what is meant by a *snake* for this sequence. **(5 marks)**

(b) Define the homomorphism $\delta\colon H_n(W) \to H_{n-1}(U)$. You should give a clear statement of the lemmas needed to ensure that your definition is meaningful, but you do not need to prove those lemmas. **(4 marks)**

(c) Suppose that $H_k(W)$ is finite for all k, and that $H_k(U) \simeq \mathbb{Z}$ for all k. Prove that $H_k(V)$ is infinite and that the map $p_*\colon H_k(V) \to H_k(W)$ is surjective. **(5 marks)**

(d) Consider the chain complex with $A_k = \mathbb{Z}^3$ for all $k \in \mathbb{Z}$ and $d(x,y,z) = (z,0,0)$.

  (i) Find the homology of A_*. **(2 marks)**

  (ii) Show that the formula $m(x,y,z) = (0,y,0)$ defines a chain map $m\colon A_* \to A_*$ **(2 marks)**

  (iii) Show that m is chain homotopic to the identity. **(3 marks)**

  (iv) Construct a chain complex A'_* where the differential is zero, and a chain homotopy equivalence from A'_* to A_*. **(4 marks)**

Solution:

(a) A snake is a list (c, w, v, u, a) where

- $c \in H_k(W)$ [1]
- $w \in Z_k(W)$ is a cycle with $c = [w]$ [1]
- $v \in V_k$ satisfies $p(v) = w$ [1]
- $u \in Z_{k-1}(U)$ satisfies $i(u) = d(v)$ [1]
- $a = [u] \in H_{k-1}(U)$. [1]

(b) It can be shown that

  (1) For any $c \in H_k(W)$, there exists a snake (c, w, v, u, a) starting with c. [1]

  (2) If we have snakes (c, w, v, u, a) and (c, w', v', u', a') both starting with c, then $a = a'$. [1]

We can therefore define $\delta\colon H_k(W)$ by $\delta(c) = a$, for any snake (c, w, v, u, a) that starts with c. [2]

(c) The Snake Lemma gives exact sequences

$$H_{k+1}(W) \xrightarrow{\delta} H_k(U) \xrightarrow{i_*} H_k(V) \xrightarrow{p_*} H_k(W) \xrightarrow{\delta} H_{k-1}(U)[1]$$

For every element c in the finite group $H_k(W)$ we know that c has finite order, so the element $\delta(c) \in H_{k-1}(U)$ also has finite order. However, $H_{k-1}(U) \simeq \mathbb{Z}$ so the only element of finite order in this group is zero. It follows that all the maps δ are zero [1], and thus that the sequence

$$H_k(U) \xrightarrow{i_*} H_k(V) \xrightarrow{p_*} H_k(W)$$

is short exact [1]. This means that p_* is surjective [1], as required. It also means that i_* is injective and $H_k(U) \simeq \mathbb{Z}$ is infinite so $H_k(V)$ must also be infinite [1].

(d) (i) We have

$$\begin{aligned} B_k(A) &= \operatorname{img}(d) = \mathbb{Z} \oplus 0 \oplus 0 \\ Z_k(A) &= \{(x,y,z) \mid (z,0,0) = (0,0,0)\} = \{(x,y,0) \mid x,y \in \mathbb{Z}\} \\ &= \mathbb{Z} \oplus \mathbb{Z} \oplus 0 \\ H_k(A) &= (\mathbb{Z} \oplus \mathbb{Z} \oplus 0)/(\mathbb{Z} \oplus 0 \oplus 0) \simeq \mathbb{Z}. \end{aligned}$$

Explicitly, we have $H_k(A) = \mathbb{Z}.h$, where $h = [(0,1,0)]$ [2].

(ii) From the formulae $d(x,y,z) = (z,0,0)$ and $m(x,y,z) = (0,y,0)$ we get $d(m(x,y,z)) = d(0,y,0) = (0,0,0)$ and $m(d(x,y,z)) = m(z,0,0) = (0,0,0)$. This shows that $dm = md$, so m is a chain map [2].

(iii) Now define $s(x,y,z) = (0,0,x)$ [1]. This has $d(s(x,y,z)) = d(0,0,x) = (x,0,0)$ and $s(d(x,y,z)) = s(z,0,0) = (0,0,z)$ so

$$(ds + sd)(x,y,z) = (x,0,z) = (\operatorname{id} - m)(x,y,z),$$

so s gives a chain homotopy between id and m [2].

(iv) Now define $A'_k = \mathbb{Z}$, with $d' = 0\colon A'_k \to A'_{k-1}$ [1]. Define $i\colon A'_k \to A_k$ by $i(y) = (0,y,0)$ [1]and $r\colon A_k \to A'_k$ by $r(x,y,z) = y$ [1]. These are chain maps with r id and $ir = m$ so ir is chain homotopic to id [1]. This means that i is a chain homotopy equivalence from A'_* to A_*.

(4) For each of the following, either give an example (with justification) or prove that no example can exist.

(a) A continuous map $f\colon X \to Y$ such that $f_*\colon H_1(X) \to H_1(Y)$ is injective but not surjective, and $f_*\colon H_{10}(X) \to H_{10}(Y)$ is surjective but not injective. **(5 marks)**

(b) A path connected space X that is homotopy equivalent to $X \times X$. **(5 marks)**

(c) A path connected space X that is not homotopy equivalent to $X \times X$. **(5 marks)**

(d) A space X and a point $x \in X$ such that X is not contractible but $X \setminus \{x\}$ is contractible. **(5 marks)**

(e) A subspace $X \subseteq \mathbb{R}^2$ that is homotopy equivalent to $S^4 \setminus S^2$. **(5 marks)**

Solution:

(a) Let $f\colon S^{10} \to S^1$ be the constant map sending all of S^{10} to the point $e_0 \in S^1$ [3]. Then $f_*\colon H_1(S^{10}) \to H_1(S^1)$ is the inclusion $0 \to \mathbb{Z}$, which is injective but not surjective [1]. Moreover, $f_*\colon H_{10}(S^{10}) \to H_{10}(S^1)$ is the zero homomorphism $\mathbb{Z} \to 0$, which is surjective but not injective [1].

(b) The spaces $I = [0,1]$ and $I \times I$ are both homotopy equivalent to a point, and thus to each other [5]. (For a more degenerate example, one could just take X to be a point.)

(c) The space S^1 is not homotopy equivalent to $S^1 \times S^1$ [3] (because $H_1(S^1) = \mathbb{Z}$ is not isomorphic to $H_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$) [2].

(d) S^1 [3] is not contractible (because $H_1(S^1) = \mathbb{Z}$ is nontrivial [1]) but $S^1 \setminus \{1\}$ is homeomorphic to $\mathbb{R}$ and thus is contractible [1].

(e) In general, $S^n \setminus S^m$ is homotopy equivalent to S^{n-m-1} [2]. In particular, the space $S^4 \setminus S^2$ is homotopy equivalent to S^1, which is a subset of $\mathbb{R}^2$ [3].

(5) Let X be a path connected space, and put

$$\begin{aligned} U &= \{(t,x) \in S^1 \times X \mid t \neq (0,1)\} \\ V &= \{(t,x) \in S^1 \times X \mid t \neq (0,-1)\}. \end{aligned}$$

We use the usual notation for inclusion maps:

$$\begin{array}{ccc} U \cap V & \xrightarrow{\ i\ } & U \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ V & \xrightarrow[\ l\ ]{} & S^1 \times X. \end{array}$$

(a) Define maps $f, g \colon X \to U \cap V$ such that f gives a homotopy equivalence from X to one path component of $U \cap V$, and g gives a homotopy equivalence from X to the other path component of $U \cap V$. **(4 marks)**

(b) Prove that the map $i' = i \circ f \colon X \to U$ is homotopic to $i \circ g$, and also that i' is a homotopy equivalence. (You can then assume without further argument that the map $j' = j \circ f \colon X \to V$ is homotopic to $j \circ g$, and that j' is a homotopy equivalence.) **(6 marks)**

(c) Deduce descriptions of the homology groups $H_p(U \cap V)$, $H_p(U)$ and $H_p(V)$, and the homomorphism

$$\alpha = \begin{bmatrix} i_* \\ -j_* \end{bmatrix} : H_p(U \cap V) \to H_p(U) \oplus H_p(V).$$

Find the kernel and image of α. **(8 marks)**

(d) Show that every element of $H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a), 0) + \alpha(b)$ for a unique pair $(a, b) \in H_p(X)^2$. **(3 marks)**

(e) Deduce that there is a short exact sequence $H_p(X) \to H_p(S^1 \times X) \to H_{p-1}(X)$. **(4 marks)**

Solution:

(a) The path components of $S^1 \setminus \{(0,1),(0,-1)\}$ are $A = [(-1,0)] = \{(x,y) \in S^1 \mid x < 0\}$ and $B = [(+1,0)] = \{(x,y) \in S^1 \mid x > 0\}$, so the path components of $U \cap V$ are $A \times X$ and $B \times X$ [2]. Here A is contractible and contains $(-1,0)$ so the map $f(x) = ((-1,0),x)$ gives a homotopy equivalence from X to $A \times X$. Similarly, the map $g(x) = ((1,0),x)$ gives a homotopy equivalence from X to $B \times X$ [2].

(b) We can define $h(t,x) = ((-\cos(\pi t), -\sin(\pi t)), x)$ for $0 \leq t \leq 1$. As $(-\cos(\pi t), -\sin(\pi t))$ lies on the bottom half of S^1, this does not pass through $(0,1) \times X$ and so gives a continuous map $[0,1] \times X \to U$. It satisfies $h(0,x) = ((-1,0),x) = i(f(x)) = i'(x)$ and $h(1,x) = ((1,0),x) = i(g(x))$, so this gives a homotopy between i' and $i \circ g$ [3]. We can also define $r \colon U \to X$ by $r(t,x) = x$. Then $r \circ i' = \mathrm{id}$, and contractibility of $S^1 \setminus \{(0,1)\}$ ensures that $i'r$ is homotopic to the identity [3].

(c) As $f \colon X \to A \times X$ and $g \colon X \to B \times X$ are homotopy equivalences, we see that every element of $H_p(U \cap V)$ can be written as $f_*(a) + g_*(b)$ for a unique pair $(a,b) \in H_p(X)^2$. [2] Similarly, any element of $H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a), j'_*(b))$ for a unique pair $(a,b) \in H_p(X)^2$[2]. As $i_* f_* = i_* g_* = i'_*$ and $j_* f_* = j_* g_* = j'_*$ we see that

$$\alpha(f_*(a) + g_*(b)) = (i'_*(a+b),\ -j'_*(a+b)\text{[2]}.)$$

This means that

$$\begin{aligned} \ker(\alpha) &= \{f_*(a) - g_*(a) \mid a \in H_p(X)\} \simeq H_p(X)\text{[1]} \\ \mathrm{img}(\alpha) &= \{(i'_*(c), -j'_*(c)) \mid c \in H_p(X)\} \simeq H_p(X)\text{[1]}. \end{aligned}$$

(d) We now see that every element $(i'_*(a), j'_*(b)) \in H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a+b), 0) + (i'_*(-b), j'_*(-b))$ with the second term lying in $\mathrm{img}(\alpha)$, and this decomposition is unique [3].

(e) From the exact sequence

$$H_p(U \cap V) \xrightarrow{\alpha} H_p(U) \oplus H_p(V) H_p(S^1 \times X) \xrightarrow{\delta} H_{p-1}(U \cap V) \xrightarrow{\alpha} H_{p-1}(U) \oplus H_{p-1}(V)$$

we get a short exact sequence

$$(H_p(U) \oplus H_p(V))/\operatorname{img}(\alpha_p) \to H_p(S^1 \times X) \to \ker(\alpha_{p-1})\textbf{[2]}$$

Part (d) gives an isomorphism $(H_p(U) \oplus H_p(V))/\operatorname{img}(\alpha_p) \simeq H_p(X)$ **[1]**. Part (c) gives an isomorphism $\ker(\alpha_{p-1}) \simeq H_{p-1}(X)$ **[1]**. We therefore have a short exact sequence

$$H_p(X) \to H_p(S^1 \times X) \to H_{p-1}(X)$$

as claimed.