The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2021–2022**

Algebraic Topology – Mock Exam — **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

1 For $n \geq 3$, we put $X_n = \mathbb{R}^2 \setminus \{(1,0),(2,0),\dots,(n,0)\}$.

(a) Define the following terms: *topology*, *topological space*, *continuous map*, *homeomorphism*. ***(7 marks)***

(b) Find a space Y_n consisting of a finite number of straight line segments that is homotopy equivalent to X_n. Give a brief justification for the claim that Y_n is homotopy equivalent to X_n. ***(6 marks)***

(c) Prove that X_n is not homeomorphic to Y_n. ***(3 marks)***

(d) Prove that X_n is not homotopy equivalent to S^m for any m. ***(4 marks)***

(e) Find contractible open sets $U_n, V_n \subseteq \mathbb{C}$ such that $X_n = U_n \cup V_n$. Give a careful proof that U_n and V_n are contractible. ***(5 marks)***

Claims about the homology of particular spaces should be stated clearly and justified briefly, but details are not required.

2 (a) Let X be a topological space. Define the equivalence relation $\sim$ on X such that $\pi_0(X) = X/\sim$, and prove that it is an equivalence relation. ***(6 marks)***

(b) Let $f\colon X \to Y$ be a continuous map. Define the induced map $f_*\colon \pi_0(X) \to \pi_0(Y)$, and prove that it is well-defined. ***(4 marks)***

(c) Show that if $f, g\colon X \to Y$ are homotopic maps then $f_* = g_*\colon \pi_0(X) \to \pi_0(Y)$. ***(4 marks)***

(d) Let Y and Z be topological spaces. Construct a bijection $\pi_0(Y \times Z) \to \pi_0(Y) \times \pi_0(Z)$, and prove that it is a bijection. ***(5 marks)***

(e) Define $i\colon \mathbb{Z} \to \mathbb{R} \setminus \mathbb{Z}$ by $i(n) = n + \frac{1}{2}$. Prove that there do not exist continuous maps $\mathbb{Z} \xrightarrow{f} S^2 \times S^2 \xrightarrow{g} \mathbb{R} \setminus \mathbb{Z}$ such that i is homotopic to $g \circ f$. ***(6 marks)***

3 (a) Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps. Define what is meant by a *snake* for this sequence. ***(5 marks)***

(b) Define the homomorphism $\delta\colon H_n(W) \to H_{n-1}(U)$. You should give a clear statement of the lemmas needed to ensure that your definition is meaningful, but you do not need to prove those lemmas. ***(4 marks)***

(c) Suppose that $H_k(W)$ is finite for all k, and that $H_k(U) \simeq \mathbb{Z}$ for all k. Prove that $H_k(V)$ is infinite and that the map $p_*\colon H_k(V) \to H_k(W)$ is surjective. ***(5 marks)***

(d) Consider the chain complex with $A_k = \mathbb{Z}^3$ for all $k \in \mathbb{Z}$ and $d(x, y, z) = (z, 0, 0)$.

(i) Find the homology of A_*. ***(2 marks)***

(ii) Show that the formula $m(x, y, z) = (0, y, 0)$ defines a chain map $m\colon A_* \to A_*$ ***(2 marks)***

(iii) Show that m is chain homotopic to the identity. ***(3 marks)***

(iv) Construct a chain complex A'_* where the differential is zero, and a chain homotopy equivalence from A'_* to A_*. ***(4 marks)***

4 For each of the following, either give an example (with justification) or prove that no example can exist.

(a) A continuous map $f\colon X \to Y$ such that $f_*\colon H_1(X) \to H_1(Y)$ is injective but not surjective, and $f_*\colon H_{10}(X) \to H_{10}(Y)$ is surjective but not injective. ***(5 marks)***

(b) A path connected space X that is homotopy equivalent to $X \times X$. ***(5 marks)***

(c) A path connected space X that is not homotopy equivalent to $X \times X$. ***(5 marks)***

(d) A space X and a point $x \in X$ such that X is not contractible but $X \setminus \{x\}$ is contractible. ***(5 marks)***

(e) A subspace $X \subseteq \mathbb{R}^2$ that is homotopy equivalent to $S^4 \setminus S^2$. ***(5 marks)***

5 Let X be a path connected space, and put

$$U = \{(t,x) \in S^1 \times X \mid t \neq (0,1)\}$$
$$V = \{(t,x) \in S^1 \times X \mid t \neq (0,-1)\}.$$

We use the usual notation for inclusion maps:

$$\begin{array}{ccc} U \cap V & \xrightarrow{\;i\;} & U \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle k} \\ V & \xrightarrow[\;l\;]{} & S^1 \times X. \end{array}$$

(a) Define maps $f, g \colon X \to U \cap V$ such that f gives a homotopy equivalence from X to one path component of $U \cap V$, and g gives a homotopy equivalence from X to the other path component of $U \cap V$. ***(4 marks)***

(b) Prove that the map $i' = i \circ f \colon X \to U$ is homotopic to $i \circ g$, and also that i' is a homotopy equivalence. (You can then assume without further argument that the map $j' = j \circ f \colon X \to V$ is homotopic to $j \circ g$, and that j' is a homotopy equivalence.) ***(6 marks)***

(c) Deduce descriptions of the homology groups $H_p(U \cap V)$, $H_p(U)$ and $H_p(V)$, and the homomorphism

$$\alpha = \begin{bmatrix} i_* \\ -j_* \end{bmatrix} : H_p(U \cap V) \to H_p(U) \oplus H_p(V).$$

Find the kernel and image of α. ***(8 marks)***

(d) Show that every element of $H_p(U) \oplus H_p(V)$ can be written as $(i'_*(a), 0) + \alpha(b)$ for a unique pair $(a,b) \in H_p(X)^2$. ***(3 marks)***

(e) Deduce that there is a short exact sequence $H_p(X) \to H_p(S^1 \times X) \to H_{p-1}(X)$. ***(4 marks)***

End of Question Paper