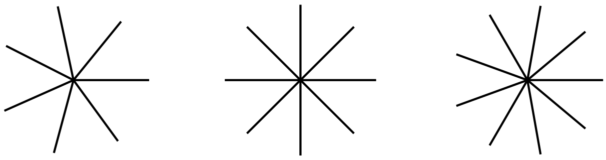
Algebraic Topology

(1)

(a) Given a topological space X, define the set $\pi_0(X)$. You should include a proof that the relevant equivalence relation is in fact an equivalence relation. **(8 marks)**

(b) Consider $[0,1]$ as a based space with 0 as the basepoint. For $n \geq 3$ we define $X_n = \{z \in \mathbb{C} \mid z^n \in [0,1]\}$:



(i) For which n and m (with $n, m \geq 3$) is X_n homotopy equivalent to X_m? **(3 marks)**

(ii) For which n and m (with $n, m \geq 3$) is X_n homeomorphic to X_m? **(4 marks)**

Justify your answers carefully.

(c) Give examples as follows, with justification:

(1) A based space W with $|\pi_1(W)| = 8$. **(3 marks)**

(2) A space X with two points $a, b \in X$ such that $\pi_1(X,a)$ is not isomorphic to $\pi_1(X,b)$. **(3 marks)**

(3) A space Y such that $H_0(Y) \simeq H_2(Y) \simeq H_4(Y) \simeq H_6(Y) \simeq \mathbb{Z}$ and all other homology groups are trivial. **(4 marks)**

Solution:

(a) We define a relation on X by declaring that $x \sim y$ if there is a continuous path $u\colon [0,1] \to X$ with $u(0) = x$ and $u(1) = y$. [1]

- For any $x \in X$ we can define $c\colon [0,1] \to X$ by $c(t) = x$ for all t. Using this we see that $x \sim x$, so or relation is reflexive. [1]
- Suppose that $x \sim y$, as witnessed by a path u from x to y. The reversed path $\overline{u}(t) = u(1-t)$ is also continuous, with $\overline{u}(0) = y$ and $\overline{u}(1) = x$, which shows that $y \sim x$. This shows that our relation is symmetric. [2]
- Suppose that $x \sim y$ and $y \sim z$, as witnessed by a path u from x to y and a path v from y to z. We can define the concatenated path $u * v\colon [0,1] \to X$ by $(u*v)(t) = u(2t)$ for $0 \leq t \leq 1/2$ and $(u*v)(t) = v(2t-1)$ for $1/2 \leq t \leq 1$ [2] (so in particular $(u*v)(1/2) = y = u(1) = v(0)$). This is continuous on the closed sets $[0,1/2]$ and $[1/2,1]$, which cover $[0,1]$, so it is continuous on $[0,1]$. As $(u*v)(0) = u(0) = x$ and $(u*v)(1) = v(1) = z$ we see that $x \sim z$. This shows that our relation is transitive. [1]

We now see that we have an equivalence relation, so we can define $\pi_0(X) = X/\sim$. [1]**[All bookwork]**

(b) (i) For any n we have a contraction of X_n to 0 given by $h(t,z) = tz$ for $0 \leq t \leq 1$. Thus, all the spaces X_n are homotopy equivalent to a point and thus to each other. [3] **[Unseen but easy]**

(ii) Note that $|\pi_0(X_n \setminus \{a\})|$ is 2 for most values of a, but it is n if $a = 0$, and 1 if $|a| = 1$. If we have a homeomorphism $f\colon X_n \to X_m$ then we get a homeomorphism $X_n \setminus \{0\} \to X_m \setminus \{f(0)\}$ so

$$n = |\pi_0(X_n \setminus \{0\})| = |\pi_0(X_m \setminus \{f(0)\})| \in \{1, 2, m\}.$$

As $n, m \geq 3$ this can only occur if $n = m$. Thus, no two of the spaces X_n are homeomorphic. [4] **[Unseen, but the general technique has been seen.]**

(c) (1) We can take $W = (\mathbb{R}P^2)^3$ [2], so $\pi_1(W) = \pi_1(\mathbb{R}P^2)^3 = (\mathbb{Z}/2)^3$, so $|\pi_1(W)| = 8$. [1]**[Unseen, but $\mathbb{R}P^2$ is a standard example.]**

(2) We can take $X = S^1 \cup \{0\} \subset \mathbb{C}$ and $a = 0$ and $b = 1$, so $\pi_1(X, a) = 0$ and $\pi_1(X, b) = \mathbb{Z}$. [3] **[Unseen]**

(3) We can take $Y = S^2 \vee S^4 \vee S^6$. This is connected, so $H_0(Y) = \mathbb{Z}$. For $i > 0$ we have $H_i(Y) = H_i(S^2) \oplus H_i(S^4) \oplus H_i(S^6)$. We also have $H_i(S^i) = \mathbb{Z}$, and $H_i(S^j) = 0$ for $j \neq i$; it follows that $H_*(Y)$ is as required. [4] Alternatively, we can take $Y = \mathbb{C}P^3$. **[Similar examples have been seen.]**

(2) Are the following true or false? Justify your answers.

(a) S^5 is a Hausdorff space. **(4 marks)**

(b) The Klein bottle is a retract of $S^1 \times S^1 \times S^1$. **(4 marks)**

(c) There is a connected space X with $\pi_1(X) \simeq \mathbb{Z}/2$ and $H_1(X) \simeq \mathbb{Z}$. **(4 marks)**

(d) There is a short exact sequence $\mathbb{Z}/9 \to \mathbb{Z}/99 \to \mathbb{Z}/11$. **(4 marks)**

(e) If K is a simplicial complex and L is a subcomplex and $H_3(K) = 0$ then $H_3(L) = 0$. **(4 marks)**

(f) If K and L are simplicial complexes and $f\colon |K| \to |L|$ is a continuous map then there is a simplicial map $s\colon K \to L$ such that f is homotopic to $|s|$. **(5 marks)**

Solution:

(a) This is true [1], because the standard topology on S^5 comes from the Euclidean metric on $\mathbb{R}^6$, and metric spaces are always Hausdorff. [3] **[It was proved in lectures that metric spaces are Hausdorff.]**

(b) This is false [1]. Let X be the Klein bottle. If this was a retract of $(S^1)^3$, then $\pi_1(X)$ would be a retract of the group $\pi_1((S^1)^3) = \mathbb{Z}^3$, so in particular it would be a subgroup of $\mathbb{Z}^3$ and so would be abelian. However, it is standard that $\pi_1(X)$ is nonabelian, so this is a contradiction. [3] **[Similar examples have been seen.]**

(c) This is false [1]. For a connected space X, the group $H_1(X)$ is always the abelianisation of $\pi_1(X)$. Thus, if $\pi_1(X)$ is $\mathbb{Z}/2$ then $H_1(X)$ must also be $\mathbb{Z}/2$. [3] **[Unseen]**

(d) This is true [1]: there is a short exact sequence $\mathbb{Z}/9 \xrightarrow{i} \mathbb{Z}/99 \xrightarrow{p} \mathbb{Z}/11$ given by $i(a \pmod 9) = 11a \pmod{99}$ and $p(b \pmod{99}) = b \pmod{11}$. [3] Alternatively, as 9 and 11 are coprime we can use the Chinese Remainder Theorem to identify $\mathbb{Z}/99$ with $\mathbb{Z}/9 \times \mathbb{Z}/11$. We then have a short exact sequence $\mathbb{Z}/9 \xrightarrow{j} \mathbb{Z}/9 \times \mathbb{Z}/11 \xrightarrow{q} \mathbb{Z}/11$ given by $j(x) = (x, 0)$ and $q(x, y) = y$. **[Similar examples have been seen.]**

(e) This is false [1]. For example, if $K = \Delta^4$ and $L = \partial\Delta^4 \subset K$ then $H_3(K) = 0$ but $H_3(L) = \mathbb{Z}$. [3] **[Seen]**

(f) This is false. [1]For example, K and L could be as follows:

If $s\colon K \to L$ is a simplicial map, it is easy to see that the image can only be a single point or a single edge of L, and thus that $|s|$ is homotopic to a constant map. However, it is easy to produce a homeomorphism $f\colon |K| \to |L|$ and then f is not homotopic to a constant, so it cannot be homotopic to $|s|$ for any s. [4] (By the Simplicial Approximation Theorem, for any $f\colon |K| \to |L|$ we can find a corresponding map $s\colon K^{(r)} \to L$ for sufficiently large r; but that is not relevant here, because the question specifies that s should be defined on K itself.) **[Similar examples have been seen.]**

(3) Let K and L be abstract simplicial complexes.

(a) Define what is meant by a *simplicial map* from K to L. **(3 marks)**

(b) Let $s, t\colon K \to L$ be simplicial maps. Define what it means for s and t to be *directly contiguous*. **(3 marks)**

(c) Prove that if s and t are directly contiguous, then the resulting maps $|s|, |t|\colon |K| \to |L|$ are homotopic. **(3 marks)**

(d) Prove that if s and t are directly contiguous, then the resulting maps $s_*, t_*\colon H_*(K) \to H_*(L)$ are the same. (You can prove the main formula just for $n = 3$ rather than general n.) **(9 marks)**

(e) How many injective simplicial maps are there from $\partial\Delta^2$ to itself? Show that no two of them are directly contiguous. **(7 marks)**

Solution:

(a) A simplicial map from K to L is a function $s\colon \mathrm{vert}(K) \to \mathrm{vert}(L)$ such that whenever $\sigma = \{v_0, \dots, v_n\}$ is a simplex of K, the image $s(\sigma) = \{\sigma(v_0), \dots, \sigma(v_n)\}$ is a simplex of L. [3]

(b) We say that s and t are directly contiguous if whenever $\sigma = \{v_0, \dots, v_n\}$ is a simplex of K, the set

$$s(\sigma) \cup t(\sigma) = \{s(v_0), \dots, s(v_n), t(v_0), \dots, t(v_n)\}$$

is a simplex of L. [3] **[Bookwork]**

(c) Suppose that s and t are directly contiguous. Consider a point $x \in |K|$, so $x \in |\sigma|$ for some $\sigma \in \mathrm{simp}(K)$. Put $\tau = s(\sigma) \cup t(\sigma)$, which is a simplex of L because of the contiguity condition. Both $|s|(x)$ and $|t|(x)$ lie in $|\tau|$, so the whole line segment from $|s|(x)$ to $|t|(x)$ lies in $|\tau|$. We can therefore define a linear homotopy $h\colon [0,1] \times |K| \to |L|$ from $|s|$ to $|t|$ by $h(r, x) = (1-r)|s|(x) + r\,|t|(x)$. [3] **[Bookwork]**

(d) Suppose again that s and t are directly contiguous. Define $u\colon C_nK \to C_{n+1}L$ by

$$u\langle v_0, \dots, v_n\rangle = \sum_{i=0}^{n} (-1)^i \langle s(v_0), \dots, s(v_i), t(v_i), \dots, t(v_n)\rangle . [2]$$

We claim that $du + ud = t_\# - s_\#$ [1]. We will prove this for a generator $x = \langle v_0, v_1, v_2, v_3\rangle \in C_3(K)$, using the abbreviated notation i for v_i or $s(v_i)$, and $\bar{i}$ for $t(v_i)$. We have

$$u(x) = +0\overline{0123} \quad -01\overline{123} \quad +012\overline{23} \quad -0123\overline{3} \qquad\qquad d(x) = +123 \;\; -023 \;\; +013 \;\; -012$$

$$du(x) = \begin{array}{cccc} +\overline{0123} & -1\overline{123} & +12\overline{23} & -123\overline{3} \\ -0\overline{123} & +0\overline{123} & -02\overline{23} & +023\overline{3} \\ +0\overline{023} & -01\overline{23} & +01\overline{23} & -013\overline{3} \\ -0\overline{013} & +01\overline{13} & -012\overline{3} & +012\overline{3} \\ +0\overline{012} & -01\overline{12} & +012\overline{2} & -0123 \end{array} \qquad ud(x) = \begin{array}{ccc} +1\overline{123} & -12\overline{23} & +123\overline{3} \\ -0\overline{023} & +02\overline{23} & -023\overline{3} \\ +0\overline{013} & -01\overline{13} & +013\overline{3} \\ -0\overline{012} & +01\overline{12} & -012\overline{2} \end{array}$$

Most terms cancel in the indicated groups, leaving $du(x) + ud(x) = \overline{0123} - 0123$. In the original notation, this says that

$$(du + ud)(x) = \langle t(v_0), t(v_1), t(v_2), t(v_3)\rangle - \langle s(v_0), s(v_1), s(v_2), s(v_3)\rangle = t_\#(x) - s_\#(x),$$

which means that u is a chain homotopy between $s_\#$ and $t_\#$ [5]. As these maps are chain-homotopic, they induce the same homomorphism between homology groups. [1]**[Bookwork]**

(f) The injective simplicial maps from $\partial\Delta^2$ to itself are just given by permuting the three vertices, so there are $3! = 6$ such maps [2]. Suppose that f and g are permutations that are contiguous. Then the set $f(\{0,1\}) \cup g(\{0,1\})$ must be a simplex, so it has size at most two. However, $f(\{0,1\})$ and $g(\{0,1\})$ both have size two already, so this is only possible if $f(\{0,1\}) = g(\{0,1\})$. As f and g are permutations, it follows that $f(2) = g(2)$. By applying the same logic to $\{0,2\}$ and then $\{1,2\}$, we also see that $f(1) = g(1)$ and $f(0) = g(0)$. Thus, we actually have $f = g$ [5]. **[Unseen]**

(4) Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps.

(a) Define what is meant by saying that the above sequence is short exact. **(3 marks)**

Now recall that a *snake* for the above sequence is a system (c, w, v, u, a) such that

- $c \in H_n(W)$;
- $w \in Z_n(W)$ is a cycle such that $c = [w]$;
- $v \in V_n$ is an element with $p(v) = w$;
- $u \in Z_{n-1}(U)$ is a cycle with $i(u) = d(v) \in V_{n-1}$;
- $a = [u] \in H_{n-1}(U)$.

(b) Prove that for each $c \in H_n(W)$ there is a snake starting with c. **(8 marks)**

(c) Prove that if two snakes have the same starting point, then they also have the same endpoint. **(10 marks)**

(d) Suppose that the differential $d\colon V_{n+1} \to V_n$ is surjective. Show that any snake starting in $H_n(W)$ ends with zero. **(4 marks)**

Solution:

(a) The map i is injective, the map p is surjective, and the image of i is the same as the kernel of p. [3] **[Bookwork]**

(b) Consider an element $c \in H_n(W)$. As $H_n(W) = Z_n(W)/B_n(W)$ by definition, we can certainly choose $w \in Z_n(W)$ such that $c = [w]$ [1]. As the sequence $U \xrightarrow{i} V \xrightarrow{p} W$ is short exact, we know that $p\colon V_n \to W_n$ is surjective, so we can choose $v \in V_n$ with $p(v) = w$ [1]. As p is a chain map we have $p(d(v)) = d(p(v)) = d(w) = 0$ (the last equation because $w \in Z_n(W)$) [1]. This means that $d(v) \in \ker(p)$, but $\ker(p) = \mathrm{img}(i)$ because the sequence is exact, so we have $u \in U_{n-1}$ with $i(u) = d(v)$ [2]. Note also that $i(d(u)) = d(i(u)) = d(d(v)) = 0$ (because i is a chain map and $d^2 = 0$) [1]. On the other hand, exactness means that i is injective, so the relation $i(d(u)) = 0$ implies that $d(u) = 0$ [1]. This shows that $u \in Z_{n-1}(U)$, so we can put $a = [u] \in H_{n-1}(U)$ [1]. We now have a snake (c, w, v, u, a) starting with c as required. **[Bookwork]**

(c) Suppose we have two snakes that start with c. We can then subtract them to get a snake $(0, w, v, u, a)$ starting with 0 [1]. It will be enough to show that this ends with 0 as well, or equivalently that $a = 0$ [1]. The first snake condition says that $[w] = 0$, which means that $w = d(w')$ for some $w' \in W_{n+1}$ [1]. Because p is surjective we can also choose $v' \in V_{n+1}$ with $w' = p(v')$ [1], and this gives $w = d(w') = d(p(v')) = p(d(v'))$ [1]. The next snake condition says that $p(v) = w$. We can combine these facts to see that $p(v - d(v')) = 0$, so $v - d(v') \in \ker(p) = \mathrm{img}(i)$[1]. We can therefore find $u' \in U_n$ with $v - d(v') = i(u')$ [1]. We can apply d to this using $d^2 = 0$ and $di = id$ to get $d(v) = i(d(u'))$ [1]. On the other hand, the third snake condition tells us that $d(v) = i(u)$. Subtracting these gives $i(u - d(u')) = 0$, but i is injective, so $u = d(u')$, so $u \in B_{n-1}(U)$ [1]. The final snake condition now says that $a = [u] = u + B_{n-1}(U)$, but $u \in B_{n-1}(U)$ so $a = [u] = 0$ [1]. **[Bookwork]**

(d) Now suppose that $d\colon V_{n+1} \to V_n$ is surjective. As $d^2 = 0$ this means that $d\colon V_n \to V_{n-1}$ is zero. Now suppose we have a snake (c, w, v, u, a) with $c \in H_n(W)$ so $v \in V_n$. The condition $i(u) = d(v)$ now gives $i(u) = 0$, but i is injective so $u = 0$, so $a = [u] = 0$. [4] **[Unseen]**

(5) Consider a simplicial complex K with subcomplexes L and M such that $K = L \cup M$. Use the following notation for the inclusion maps:

$$\begin{array}{ccc} L \cap M & \xrightarrow{i} & L \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow[g]{} & K. \end{array}$$

(a) State the Seifert-van Kampen Theorem (in a form applicable to simplicial complexes and subcomplexes as above). **(4 marks)**

(b) State the Mayer-Vietoris Theorem. **(5 marks)**

(c) State a theorem about the relationship between π_1 and H_1. **(3 marks)**

(d) Suppose that $|L|$, $|M|$ and $|L \cap M|$ are all homotopy equivalent to S^1. Suppose that the maps i and j both have degree two.

(1) Find a presentation for $\pi_1|K|$. **(3 marks)**

(2) Find $H_*(K)$. In particular, you should express each nonzero group as a direct sum of terms like $\mathbb{Z}$ or $\mathbb{Z}/n$. **(10 marks)**

Solution:

(a) Suppose that $|L \cap M|$ is connected and that we have presentations

$$\begin{aligned} \pi_1|L| &= \langle x_1, \dots, x_p \mid u_1 = \dots = u_k = 1 \rangle \\ \pi_1|M| &= \langle y_1, \dots, y_q \mid v_1 = \dots = v_l = 1 \rangle \\ \pi_1|L \cap M| &= \langle z_1, \dots, z_r \mid w_1 = \dots = w_m = 1 \rangle. \end{aligned}$$

Then we have a presentation of $\pi_1|K|$ with generators $x_1, \dots, x_p, y_1, \dots, y_q$ and relations $u_1 = \dots = u_r = v_1 = \dots = v_l = 1$ and $i_*(z_t) = j_*(z_t)$ for all t. [4] **[Bookwork]**

(b) There is a natural map $\delta \colon H_n(K) = H_n(L \cup M) \to H_{n-1}(L \cap M)$ such that the resulting sequence

$$H_{n+1}(L \cup M) \xrightarrow{\delta} H_n(L \cap M) \xrightarrow{\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]} H_n(L) \oplus H_n(M) \xrightarrow{[\, f_* \; g_* \,]} H_n(L \cup M) \xrightarrow{\delta} H_{n-1}(L \cap M)$$

is exact for all n [5]. **[Bookwork]**

(c) If $|K|$ is connected [1], then $H_1(K)$ is naturally isomorphic to the abelianisation of $\pi_1|K|$ [2]. **[Bookwork]**

(d) (1) As $|L \cap M| \simeq S^1$, we can choose a generator z for $\pi_1|L \cap M|$. As i has degree two we see that there is a generator x of $\pi_1|L|$ with $i_*(z) = x^2$. As j has degree two we see that there is a generator y of $\pi_1|M|$ with $j_*(z) = y^2$. The Seifert-van Kampen Theorem now gives $\pi_1|K| = \langle x, y \mid x^2 = y^2 \rangle$. [3] **[Similar examples have been seen.]**

(2) We have a Mayer-Vietoris sequence as follows:

$$\begin{array}{l} H_2(L \cap M) \xrightarrow{\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]} H_2(L) \oplus H_2(M) \xrightarrow{[\, f_* \; g_* \,]} H_2(K) \to \\ \to H_1(L \cap M) \xrightarrow{\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]} H_1(L) \oplus H_1(M) \xrightarrow{[\, f_* \; g_* \,]} H_1(K) \to \\ \to H_0(L \cap M) \xrightarrow{\left[\begin{smallmatrix} i_* \\ -j_* \end{smallmatrix}\right]} H_0(L) \oplus H_0(M) \xrightarrow{[\, f_* \; g_* \,]} H_0(K). \end{array}$$ [3]

The spaces $|L \cap M|$, $|L|$ and $|M|$ are all homotopy equivalent to S^1 and so have $H_0 = H_1 = \mathbb{Z}$ and all other homology groups are zero. We also know that i_* and j_* act as the identity on H_0, and as multiplication by 2 on H_1. The sequence therefore has the following form:

$$\begin{array}{l} 0 \xrightarrow{0} 0 \xrightarrow{0} H_2(K) \to \\ \to \mathbb{Z} \xrightarrow{\left[\begin{smallmatrix} 2 \\ -2 \end{smallmatrix}\right]} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{[\, f_* \; g_* \,]} H_1(K) \to \\ \to \mathbb{Z} \xrightarrow{\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{[\, f_* \; g_* \,]} H_0(K). \end{array}$$ [3]

From this we can read off that $H_2(K) = 0$ and $H_0(K) = \mathbb{Z}$ [1]and that $H_1(K) = \mathbb{Z}^2/\mathbb{Z}.(2, -2)$ [1]. If we use the basis $\{(1, 0), (1, -1)\}$ for $\mathbb{Z}^2$ we get $H_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z}/2$ [1]. By extending the sequence further upwards, it is also clear that $H_n(K) = 0$ for $n > 2$ [1]. **[Similar examples have been seen.]**