MAS435, Algebraic Topology

2017–18 Exam

Solutions

All Q1 is bookwork except for Part (d). Which is unseen.

A1: (i) (a) A topological space is a set X together with a collection $\mathcal{T}$ of subsets (called open sets) so that

Top 1: $\emptyset, X \in \mathcal{T}$

Top 2: $U_i \in \mathcal{T} \Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}$ 3

Top 3: $U, V \in \mathcal{T} \Rightarrow U \cap V \in \mathcal{T}$

If $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ are topological spaces the product topology on $X \times Y$ consists of subsets which are unions of those of the form $U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$ 2

(b) Suppose $f: T \longrightarrow X \times Y$ & write $f_X = \pi_X \circ f$

$f_Y = \pi_Y \circ f$

and that f_X, f_Y are continuous. First note that if $U \in \mathcal{T}_X$, $V \in \mathcal{T}_Y$ then $f_X^{-1}(U)$, $f_Y^{-1}(V)$ are open

& $$f^{-1}(U \times V) = f_X^{-1}(U) \cap f_Y^{-1}(V).$$

so $f^{-1}(U \times V)$ is open by Top3.

Now if $W = \bigcup_i U_i \times V_i$, $f^{-1}(W) = \bigcup_i f^{-1}(U_i \times V_i)$

is open by Top 2. 4

(ii) (a) A path from a to b in X is a continuous function $f:[0,1]\longrightarrow X$ with $f(0)=a$, $f(1)=b$ 2

(b) The concatenated path is defined by

$$(\omega\circ\sigma)(t)=\begin{cases}\omega(2t) & t\in[0,\tfrac{1}{2}]\\ \sigma(2t-1) & t\in[\tfrac{1}{2},1]\end{cases}$$ 2

This is well defined since $\omega(2\cdot\tfrac{1}{2})=b=\sigma(2\cdot\tfrac{1}{2}-1)$ & continuous by the gluing lemma.

To see $\sim$ is an equivalence relation we need to prove three things,

reflexive: The constant path at x shows $x\sim x$.

symmetric: If $a\sim b$ & ω is a path from a to b then the reverse path $\overline{\omega}(t)=\omega(1-t)$ is a path from b to a showing $b\sim a$ 3

transitive: If $a\sim b$, $b\sim c$ by the path ω, σ then $a\sim c$ by the path $\omega\circ\sigma$.

Now take $\pi_0(X)=X/\sim$

(c) Define $f_*:\pi_0(X)\longrightarrow\pi_0(Y)$ by $[x]\longmapsto[f(x)]$ 3

This is well defined since if ω is a path from a to b

then $f \circ \omega$ is a path from $f(a)$ to $f(b)$.

(d) We define a map $\theta : \pi_0(X \times Y) \longrightarrow \pi_0(X) \times \pi_0(Y)$ by giving it components $(\pi_X)_*$ & $(\pi_Y)_*$ 2

We define a map $\varphi : \pi_0(X) \times \pi_0(Y) \longrightarrow \pi_0(X \times Y)$ by

$$([x], [y]) \longmapsto [(x,y)]$$ 2

This is well defined since if ω is a path from x to x', σ ——— y to y', then $\{\omega, \sigma\} : [0,1] \longrightarrow X \times Y$ is a path ~~from~~ (x,y) to (x', y').

We immediately check

$$(\varphi \circ \theta)([(x,y)]) = \varphi([x],[y]) = [(x,y)]$$

$$(\theta \circ \varphi)([x],[y]) = \theta([(x,y)]) = ([x],[y])$$ 2

so θ & φ are inverse bijections.

Part (i) is seen. Part (ii) is very similar to seen. Part (iii) is un seen.

A2: (i)(a) A map $p : \tilde{X} \longrightarrow X$ is a covering map if every point $x \in X$ has an open neighbourhood U so that $p^{-1}U = \coprod_\alpha \tilde{U}_\alpha$ & $p|_{\tilde{U}_\alpha} : \tilde{U}_\alpha \xrightarrow{\cong} U$ is a homeo 4

(b) The path lifting lemma states that given a path ω from x_0 in X and a point $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) = x_0$ there is a unique path $\tilde{\omega}$ from $\tilde{x}_0$ in $\tilde{X}$ with $p \circ \tilde{\omega} = \omega$.

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ {[0,1]} \arrow[r, "\omega"] \arrow[ur, dashed, "\tilde{\omega}"] & X \end{tikzcd}$$ 3

We define $\ell : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0)$

by $[\omega] \longmapsto \tilde\omega(1)$ 3

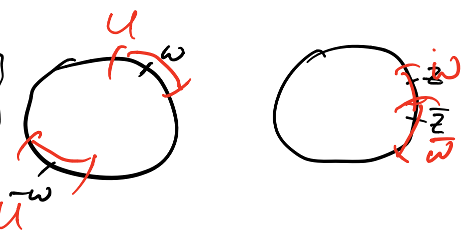
ℓ is surjective if $\tilde X$ is path connected 2

ℓ is injective if $\pi_1(\tilde X, \tilde x_0) = 1$.

(ii)(a) If $k \in K$ then $p^{-1}(k) = \{(w, z), (-w, \bar z)\}$ for some w, z, say $w = e^{2\pi i\theta}$

Choose $\tilde V = \{e^{2\pi i\theta'} \mid |\theta' - \theta| < \frac{\pi}{2}\}$

& $\tilde W$ any neighbourhood of z



Then $U = p(\tilde V \times \tilde W)$ is a neighbourhood of k with

$p^{-1}U = \tilde V \times \tilde W \sqcup (-\tilde V, \overline{\tilde W})$ & $p : \tilde V \times \tilde W \xrightarrow{\cong} U$

$p : (-\tilde V) \times \overline{\tilde W} \xrightarrow{\cong} U$. 4

(b) Suppose $[\tilde\omega] \in \pi_1(T, x_0)$ has $p_*([\tilde\omega]) = e$.

Then we have a loop homotopy $H : \omega \simeq c_{x_0}$. (where $\omega = p \circ \tilde\omega$)

Now by the Homotopy Lifting Lemma there is a homotopy $\tilde H$ starting at $\tilde\omega$ and lying over H. $\tilde H$ necessarily ends at $c_{\tilde x_0}$ since $\tilde H|_{1\times[0,1]}$ is a path in $p^{-1}(x_0)$ starting at $\tilde x_0$ & so $[\tilde\omega] = [c_{\tilde x_0}] = e$. 3

$$\begin{tikzcd} & \tilde X \arrow[d] \\ {[0,1]^2} \arrow[ur, dashed, "\tilde H"] \arrow[r, "H"'] & X \end{tikzcd}$$

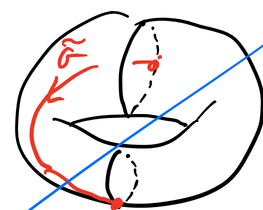
Suppose $g = [\omega]$ & let $\tilde\omega$ be a lift of ω from $\tilde x_0$, as guaranteed by the PLL.

Thus $\tilde\omega(1) \in \{\tilde x_0, f(\tilde x_0)\}$. If $\tilde\omega(1) = \tilde x_0$ then $\tilde\omega$ is a loop & $g = p_*[\tilde\omega]$ 2

If $\tilde\omega(1) = f(\tilde x_0)$ then $\tilde\omega \cdot \tilde\sigma$ is a loop in T
& $p_*[\tilde\omega\cdot\tilde\sigma] = [(p\circ\tilde\omega)\cdot(p\circ\tilde\sigma)] = g\cdot h$

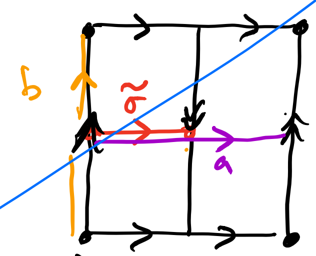
Since $\pi_1(T) \cong \mathbb{Z}\times\mathbb{Z}$, there are generators α, β of $\pi_1(T)$ & the image of p_* is generated by $p_*(\alpha), p_*(\beta)$.
Hence $\pi_1(K)$ is generated by $\{p_*(\alpha), p_*(\beta), h\}$. 3.

(c) Now take $\tilde\sigma$ to be given by $\tilde\sigma(t) = (e^{i\pi t}, 1)$



Then h^2 lifts to the loop
$$a: t \longmapsto (e^{2\pi i t}, 1)$$
we may take as $\alpha = [a]$.
Thus $p_*\alpha = h^2$ & $\pi_1(K)$ is generated by β & h 2

we see from the picture that $\bar{\tilde\sigma}\cdot b\cdot\tilde\sigma \simeq \bar b$



ie $h^{-1} p_*(\beta)\, h = (p_*\beta)^{-1}$ 2
so h does not commute with $p_*(\beta)$.

Parts (a) & (ii)(a) are bookwork. This method for $H_*((\Delta^n)^{(d)})$ has been seen for $d = n-1$, but for $d = n-2$ it is unseen.

A3: (i)(a) A chain complex of abelian groups is a sequence of ab groups & group homomorphisms

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$ 3

so that

$$d_n \circ d_{n+1} = 0$$

for all n.

The n^{th} homology of the above complex is

$$H_n(C_\bullet) = \frac{\ker(C_n \xrightarrow{d_n} C_{n-1})}{\mathrm{im}(C_{n+1} \xrightarrow{d_{n+1}} C_n)}.$$ 2

(ii)(a) If K is a d-dimensional simplicial complex, the chain complex $C_\bullet K$ takes the form

$$\to 0 \to 0 \to C_d K \to C_{d-1} K \to \cdots$$

(ie $C_i K = 0$ for $i > d$)

Hence $$H_d K = \frac{\ker(C_d K \to C_{d-1} K)}{\mathrm{im}(0 \to C_d K)} = \ker(C_d K \to C_{d-1} K)$$ 3

is a finitely generated subgroup of the free abelian group $C_d K$, & is therefore free abelian. 1

If L contains all the simplices of K of dimension $< d$ then $C_j L = C_j K$ for $j \leq d$.

Hence if $i \leq d-1$

$$H_i L = \frac{\ker(C_i L \to C_i(K))}{\mathrm{im}(C_{i+1} L \to C_i(K))} = \underbrace{\frac{\ker(C_i K \to C_{i-1} K)}{\ker(C_{i+1} K \to C_i K)}}_{H_i K \,/\!/} $$ 3

(b) Take $L = (\Delta^n)^{(d)}$, $K = \Delta^n$. & $H_* \Delta^n = H_*(pt)$. 2

Certainly $H_i L = 0$ for $i \geq d+1$ since L has no simplices of dimension $\geq d+1$. 1

By (a) $H_i L = H_i K$ for $i < d$.

Hence for $n \geq 2$ $H_0 L = \mathbb{Z}$ 2

$H_i L = 0$ for $i \neq d$.

Finally we consider H_d.

$$H_d((\Delta^n)^{(d)}) = \ker(C_d((\Delta^n)^{(d)}) \to C_{d-1}((\Delta^n)^{(d)})$$

$$= \ker(C_d(\Delta^n) \to C_d(\Delta^n)) \quad 2$$

$$= \operatorname{im}(C_{d+1}(\Delta^n) \xrightarrow{d_n} C_d(\Delta^n))$$

(since $H_d \Delta^n = 0$).

<u>Case $d = n-1$:</u> $\underset{\mathbb{Z}}{\underset{\parallel}{C_n(\Delta^n)}} \rightarrowtail C_{n-1}(\Delta^n)$ is injective since $H_n \Delta^n = 0$.

$\therefore H_{n-1}((\Delta^n)^{(n-1)}) = \mathbb{Z}$ 1

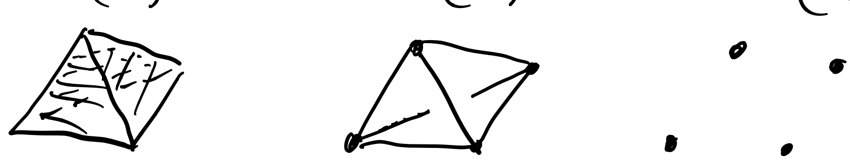
<u>Case $d = n-2$:</u> $0 \to \underset{\mathbb{Z}}{\underset{\parallel}{C_n \Delta^n}} \to \underset{\mathbb{Z}^{n+1}}{\underset{\parallel}{C_{n-1} \Delta^n}} \longrightarrow C_{n-2} \Delta^n$

3

Since the boundary of Δ^n is a generator of $\mathbb{Z}^{n+1}$ we conclude

$$H_{n-2}((\Delta^n)^{(n-2)}) = C_{n-1}\Delta^n / C_n(\Delta^n) = \mathbb{Z}^{n+1}/\mathbb{Z} \cong \mathbb{Z}^n.$$

$(\Delta^3)^{(2)}$ $(\Delta^3)^{(1)}$ $(\Delta^3)^{(0)}$



2

$(\Delta^3)^{(d)} = \Delta^3$ for $d \geq 3$.

Part (i) Bookwork. The facts about the Möbius strip are seen. This particular M–V sequence is unseen.

4. (i) If $K = L \cup M$ (union of simplicial complexes) then there is a natural long exact sequence

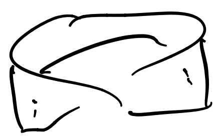
$$\cap \longrightarrow \oplus \longrightarrow \cup$$

$$\hookrightarrow H_n(L\cap M) \xrightarrow{\{(j_L)_*, -(j_M)_*\}} H_n(L) \oplus H_n(M) \xrightarrow{\langle (i_L)_*, (i_M)_* \rangle} H_n(L \cup M) \xrightarrow{\partial} H_{n-1}(L \cap M) \longrightarrow$$ 3

where the maps are induced by the inclusions.

\begin{tikzcd} & K = L \cup M & \\ L \arrow[ur, "i_L"] & & M \arrow[ul, "i_M"'] \\ & L\cap M \arrow[ul, "j_L"] \arrow[ur, "j_M"'] & \end{tikzcd} 2

$|L| = T^2$ with $H_i(T^2) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}\oplus\mathbb{Z} & 1 \\ \mathbb{Z} & 2. \end{cases}$ 4

$|M| = M\ddot{o} =$  $\simeq S^1$, $|L \cap M| \cong S^1$. $H_i(S^1) = \mathbb{Z}$ $i = 0,1$

$$\cap \qquad \mathbb{Z} \cong \oplus \qquad \cup$$

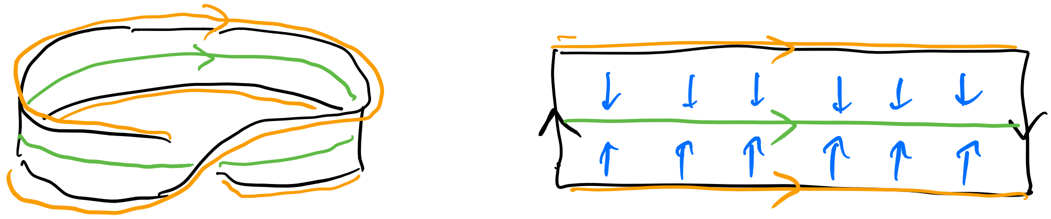
2 $\hookrightarrow H_2(\cancel{L\cap M})_0 \longrightarrow H_2(L) \oplus H_2(\cancel{M})_0 \longrightarrow H_2(L\cup M) \to$

1 $\hookrightarrow H_1(L\cap M) \cong \mathbb{Z} \xrightarrow{\alpha} H_1(L) \oplus H_1(M) \cong \mathbb{Z}^2 \oplus \mathbb{Z} \longrightarrow H_1(L \cup M) \to$ 4

0 $\hookrightarrow H_0(L\cap M) \cong \mathbb{Z} \xrightarrow{\beta} H_1(M) \oplus H_1(M) \cong \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(L\cup M) \to 0$

The maps in H_0 are easily identified so $H_0(X) \cong \mathbb{Z}$ & β is injective.

Note that j_m is inclusion of the boundary circle of Mö into Mö so it is multiplication by 2 in H_1.



To calculate $H_*(\text{Mö})$ note $\text{Mö} \simeq S^1_m$ (the green meridional circle) by the illustrated projection.

Under this, the generator of $H_1(S^1_\partial)$ (the orange cycle) maps to twice the green generator. [The subdivisions are omitted for clarity.

4

In particular α is injective so

$$H_2(X) \cong H_2(L) \cong \mathbb{Z}.$$

This means we have

$$0 \to H_1(L \cap M) \xrightarrow{\alpha} H_1(L) \oplus H_1(M) \to H_1(X) \to 0$$

$$\begin{array}{ccc} H_1(L\cap M) & & H_1(L)\oplus H_1(M) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} & \longrightarrow & (\mathbb{Z} \oplus \mathbb{Z}) \oplus \mathbb{Z} \\ 1 & \longmapsto & ((a,b),2) \end{array}$$

4

Suppose (a,b) is m times a generator

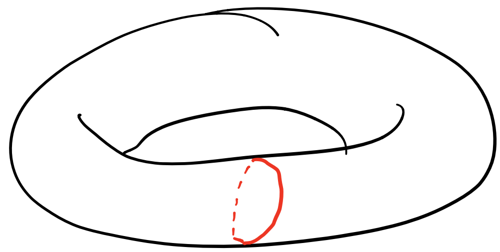
if m is odd $H_1(X) \cong \mathbb{Z}^2$

if m is even $H_1(X) \simeq \mathbb{Z}^2 \oplus \mathbb{Z}/2$.

Both can occur. If Mö is stuck by a small disc (contractible in T^2) then $m=0$

4

If Mö is stuck around the illustrated curve then $(a,b)=(1,0)$



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5. (a) True. [Each part: 1 mark for correct / Never more than 2 for incorrect] Bookwork

This is the Brouwer fixed point theorem.

If $f:\overline{B^3} \longrightarrow \overline{B^3}$ has no fixed point, we may define $\varphi: \overline{B^3} \longrightarrow S^2$

$P \longmapsto$ Intersection of the ray $P \to f(P)$ with S^2

& get

$$\begin{tikzcd} S^2 \arrow[r, hook, "i"] \arrow[dr, "id"'] & \overline{B^3} \arrow[d, "\varphi"] \\ & S^2 \end{tikzcd}$$

giving

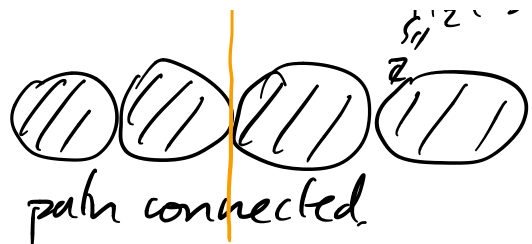
$$\begin{tikzcd} H_2(S^2) \arrow[r] \arrow[dr, "\cong"] & H_2(\overline{B^3}) \arrow[d] \\ & H_2(S^2) \end{tikzcd}$$

$H_2(S^2) \cong \mathbb{Z}$, $H_2(\overline{B^3}) \cong 0$

$x = \frac{1}{2}$

which is a contradiction.

(ii) False [Unseen example]



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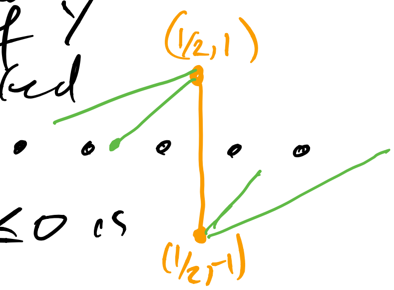
The space X is not path connected.
For example any path from $(0,0)$ to $(1,0)$ must have x coordinate $\frac{1}{2}$ at some time (Intermediate Value Theorem)
But there is no point $(\frac{1}{2}, y)$ in the space.
On the other hand Y is path connected.
Every point (x,y) of Y with $y \geq 0$ is connected $(\frac{1}{2},1)$ by a straight line
Every point with $y \leq 0$ is connected to $(\frac{1}{2}, -1)$



These two points are connected by a straight line path. 5

(iii) False. [Unseen example]

If we have a covering $\tilde{X} \xrightarrow{P} X$ with both spaces connected then $\pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{P_*} \pi_1(X, x_0)$ is injective. (by the homotopy lifting lemma).
However $\pi_1(K)$ is nonabelian whilst $\pi_1(T) \cong \mathbb{Z} \oplus \mathbb{Z}$ is abelian. 5

(iv) False. [Putting together various seen facts]

$\mathbb{R}^2 \setminus n$ points is homotopy equivalent to a bouquet of n circles. This is not equivalent to $\mathbb{R}P^2$
(by π_1: S^1 is a retract of $\bigvee^n S^1$ so $\pi_1(\mathbb{R}^2 \setminus n\text{pts})$ is infinite, whilst $\pi_1 \mathbb{R}P^2$ is of order 2

by H_1: $H_1(\mathbb{R}^2 \setminus n\text{pts}) \cong \mathbb{Z}^n$

by H_1: $H_1($[illegible]$) \cong \mathbb{Z}$

$H_1(\mathbb{RP}^2) \cong \mathbb{Z}/2.$)

5

(v) True [Seen]

$$\mathbb{R}^3 \setminus z\text{ axis} = \left(\mathbb{R}^2 \setminus \{(0,0)\}\right) \times \mathbb{R} \simeq \mathbb{R}^2 \setminus \{(0,0)\}$$

$$\simeq S^1.$$

5