MAS435 Exam 16-17 – SOLUTIONS

1.(i)(a) Bookwork A **topological space** is a set X, together with a set $\mathcal{J}$ of subsets of X so that ① $\emptyset, X \in \mathcal{J}$ 3

② $A, B \in \mathcal{J} \Rightarrow A \cap B \in \mathcal{J}$

③ $A_i \in \mathcal{J}$ for $i \in I \Rightarrow \bigcup_i A_i \in \mathcal{J}$

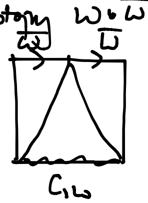
Let $\pi: X \to Y$ be the quotient map.
The quotient topology on $\mathcal{J}$ on Y is the collection of subsets $V \subseteq Y$ so that $\pi^{-1}(V) \in \mathcal{J}$. 2

(b) Bookwork If f is continuous then $f \circ \pi$ is continuous (being the composite of two continuous maps)
Conversely if $\pi \circ f$ is continuous & $V \subseteq Y$ is open then $(f \circ \pi)^{-1}(V) = \pi^{-1}(f^{-1}(V))$ is open. 4
But this means (by def^n of the quotient topology) that $f^{-1}(V)$ is open as required.

If X is compact and $\mathcal{V} = \{V_\alpha\}$ is an open cover of Y then $\pi^{-1}\mathcal{V} = \{\pi^{-1}(V_\alpha)\}$ is an open cover of X & hence has a finite subcover $\{f^{-1}(V_{\alpha_1}), \dots, f^{-1}(V_{\alpha_N})\}$. Hence $\{V_{\alpha_1}, \dots, V_{\alpha_N}\}$ is an open cover of Y. 2
The converse is false. For example the indiscrete equivalence relation has $Y = pt$ (which is compact), whilst there exist non compact spaces X (eg $X = \mathbb{R}$) 1

(ii)(a) Bookwork A loop based at x_0 is a continuous function $\omega: [0,1] \to X$ with $\omega(0) = \omega(1) = x_0$ 3
(b) The reverse of ω is $\overline{\omega}$ def by $\overline{\omega}(s) = \omega(1-s)$
The concatenated loop $\omega \cdot \sigma$ is defined by
$$(\omega \cdot \sigma)(s) = \begin{cases} \omega(2s) & 0 \leq s \leq \frac{1}{2} \\ \sigma(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$ 3

A homotopy $\omega \cdot \overline{\omega} \simeq c_{x_0}$ is defined by



$$H: [0,1] \times [0,1] \to X$$
$$(s,t) \longmapsto \begin{cases} \omega(2s) & 0 \leq s \leq \frac{t}{2} \\ \omega(t) & t/2 \leq s \leq 1 - t/2 \\ \overline{\omega}(2s-1) & 1 - \frac{t}{2} \leq s \leq 1 \end{cases}$$ 4

(c) We define $f: S^1 \to X$ by $e^{2\pi i s} \mapsto \omega(s)$ & $\lambda: [0,1] \to S^1$ by $s \mapsto e^{2\pi i s}$ Unseen.
It is then immediate that $f \circ \lambda = \omega$ & $f_*[\lambda] = [f \circ \lambda] = [\omega]$
Any group homomorphism takes the identity to the identity so if $f_*([\lambda]) = [\omega] \neq e$ then $[\lambda] \neq e$ & so $\pi_1(S^1) \not\cong 1$ 4

Q2: (a) Bookwork A covering map is a continuous function $p: \tilde{X} \to X$ so that every point $x \in X$ has a neighbourhood $U \subseteq X$ so that $p^{-1}(U) \cong \coprod_\alpha \tilde{U}_\alpha$ & $p|_{\tilde{U}_\alpha}: \tilde{U}_\alpha \xrightarrow{\cong} U$ 1

4

is a homeomorphism

Bookwork

(b) Given a path $\omega : [0,1] \longrightarrow X$ from x_0 to x_1 & a point $\tilde{x}_0 \in \tilde{X}$ with $p(\tilde{x}_0) = x_0$ there is a unique $\tilde{\omega} : [0,1] \longrightarrow \tilde{X}$ so that $\tilde{\omega}(0) = \tilde{x}_0$ & $p \circ \tilde{\omega} = \omega$. 4

we choose $\tilde{x}_0$ with $p(\tilde{x}_0) = x_0$ &

To define $\ell : \pi_1(X, x_0) \longrightarrow p^{-1}(x_0)$ by $p([\omega]) = \tilde{\omega}(1)$ where $\tilde{\omega}$ is the unique lift of ω to a path from $\tilde{x}_0$. 2

ℓ is surjective if $\tilde{X}$ is path connected 2

ℓ is injective if $\pi_1(\tilde{X}, \tilde{x}_0) = 1$.

Unseen

(ii)(a) For each pair i, j since Y is Hausdorff we may find open sets U_{ij}, U_{ji} with $U_{ij} \cap U_{ji} = \emptyset$ & $y_i \in U_{ij}$, $y_j \in U_{ji}$

Now take $U_i = \bigcap_j U_{ij}$ which is open because the intersection is finite. 2

We have $y_i \in U_i$ & $U_i \cap U_j \subseteq U_{ij} \cap U_{ji} = \emptyset$.

Unseen.

(b) Given any $x \in X$ we note x is the orbit $\{gy \mid g \in G\}$ of $y \in Y$. 1

Since Y is Hausdorff we may choose disjoint neighbourhoods $V_g \ni gy$ 2

& take $V = \bigcap_{g \in G} g^{-1} V_g$. 1

Now take $U = q(V)$ & observe it is evenly covered since $q^{-1} U = q^{-1} q V = \coprod_g gV$. 2

(c) Since G is a subgroup of $SU(n)$ it acts freely (since $gh = h$ implies $g = ghh^{-1} = hh^{-1} = e$) 1

Hence Unseen.

$SU(n) \longrightarrow SU(n)/G$ is a covering by the above. 2

Since $SU(n)$ is simply connected $\ell : \pi_1(SU(n)/G) \xrightarrow{\cong} p^{-1}p(e) = G$ is a bijection 1

It was proved in lectures that it also preserves the group operation. 1

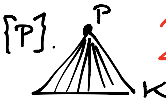
A3: (i) A chain complex is a sequence $\cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \xrightarrow{d_{n-1}}$ of abelian groups & group homs so that $d_{n-1} \circ d_n = 0$ for all n Bookwork 2

The homology of a chain complex $(A_\cdot, d)$ is $H_n(A_\cdot, d) = \dfrac{\ker(d_n : A_n \longrightarrow A_{n-1})}{\operatorname{im}(d_{n+1} : A_{n+1} \longrightarrow A_n)}$. 2

Mayer-Vietoris Theorem: If $K = L \cup M$ (union of simplicial complexes) then there is a long exact sequence

$$\begin{array}{lcl}
 & \cap \longrightarrow \oplus \longrightarrow \cup & \\
n+1 & \longrightarrow H_{n+1}(L) \oplus H_{n+1}(M) \xrightarrow{\langle i_{L*}, i_{M*}\rangle} H_{n+1}(L \cup M) \xrightarrow{\partial} & \\
n & H_n(L \cap M) \xrightarrow{[j_{L*}, -j_{M*}]} H_n(L) \oplus H_n(M) \xrightarrow{\langle i_{L*}, i_{M*}\rangle} H_n(L \cup M) \xrightarrow{\partial} & 2 \\
n-1 & H_{n-1}(L \cap M) &
\end{array}$$

$$\begin{array}{ccccc}
 & & K & & \\
 & {\scriptstyle i_L}\nearrow & & \nwarrow{\scriptstyle i_M} & \\
L & & 1 & & M \\
 & {\scriptstyle j_L}\nwarrow & & \nearrow{\scriptstyle j_M} & \\
 & & L \cap M & &
\end{array}$$

(11)(a) The $\underline{P\text{-cone}}$ on K is defined by $C_pK = K \cup \{[P,\sigma] \mid \sigma \in K\} \cup [P]$.  2

Bookwork

We show there is a chain homotopy $h: C_n(C_pK) \to C_{n+1}(C_pK)$

$$\sigma \longmapsto \begin{cases} P\sigma & P \notin \sigma \\ 0 & P \in \sigma. \end{cases} \qquad \left(\begin{array}{l}\text{if } \sigma = \langle v_0, \dots, v_n\rangle \\ \text{then } P\sigma = \langle P, v_0, \dots, v_n\rangle\end{array}\right)$$ 2

We claim $h: id \simeq \varepsilon_p$ ie $hd + dh = id - \varepsilon_p$ where $\varepsilon_p : C_n(C_pK) \to C_n(C_pK)$ is zero unless $n=0$ when $\langle v\rangle \longmapsto \langle P\rangle$

From degree 0 we have $(hd+dh)\langle v\rangle = dh\langle v\rangle = \begin{bmatrix} 0 & \text{if } v=P \\ d\langle P,v\rangle & \text{if } v\neq P \end{bmatrix} = \begin{bmatrix} \langle P\rangle - \langle P\rangle & \text{if } v=P \\ \langle v\rangle - \langle P\rangle & \text{if } v \neq P\end{bmatrix} = (1-\varepsilon_p)\langle v\rangle$

If $P \notin \sigma$ we find

$$(hd+dh)(\sigma) = h(d\sigma) + d(P\sigma) = Pd\sigma + \sigma - Pd\sigma = \sigma$$ 2

If $P \in \sigma$ we find

$$(hd+dh)(P\sigma) = h(d(P\sigma)) + 0 = h(\sigma - Pd\sigma) = P\sigma - 0 = P\sigma$$ 2

Since $id \simeq \varepsilon_p$, we see $C_*(C_pK) \simeq C_*(P)$ & $H_*(C_pK) \cong H_*(P)$ as required. 1

(b) Let $K = \Sigma L = c_N L \cup c_S L$. Thus $c_N L \simeq N$ & $c_S L \simeq S$ & $c_N L \cap c_S L = L$ 2

[Unseen]

thus the Mayer-Vietoris sequence reads

$$H_{i+1}(c_N L) \oplus H_{i+1}(c_S L) \to H_{i+1}(\Sigma L) \xrightarrow{\partial}$$
$$\hookrightarrow H_i(c_N L \cap c_S L) \to H_i(c_N L) \oplus H_i(c_S L) \to$$ 2

If $i \geq 1$ $H_i(c_N L) = H_{i+1}(c_N L) = 0$ & similarly for $c_S L$ 2

hence ∂ is an iso

$$H_{i+1}(\Sigma L) \cong H_i(L) \text{ as required.}$$

A4 (i)(a) $\Lambda(\theta) = \sum_{i=0}^{\infty} (-1)^i \operatorname{trace}(\theta : C_i \to C_i)$ Bookwork 2

b) We have $C_n \supseteq Z_n \supseteq B_n$ & we may choose basis $b_n^1, \dots, b_n^{r_n}$ of B_n

Bookwork

& extend by $z_n^1, \dots, z_n^{s_n}$ to give a basis of Z_n

& extend by $c_n^1, \dots, c_n^{t_n}$ to give a basis of C_n 2

Furthermore we choose these bases for decreasing n, starting with the top nonzero one.

Then we take

$$b_n^i = d(c_{n+1}^i) \quad \& \ \theta_n \text{ has matrix } \quad \Theta_n = \left(\begin{array}{c|c|c} P_n & ? & ? \\ \hline ? & Q_n & ? \\ \hline ? & ? & R_n \end{array}\right)$$ 2

Thus $tr(\Theta_n) = tr(P_n) + tr(Q_n) + tr(R_n)$

& since Θ_n is a chain map

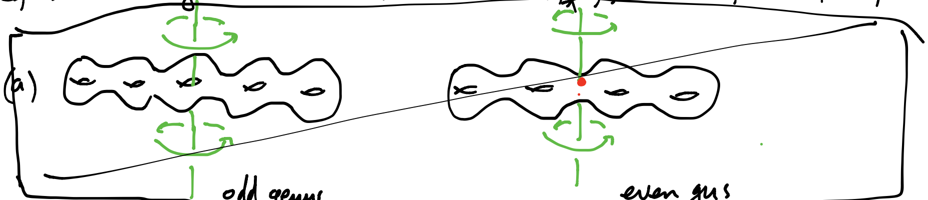
$$R_{n+1}' = P_n .$$

Hence $\Lambda(\theta) = \sum(-1)^i \mathrm{tr}(\theta_i) = \sum(-1)^i \left[\mathrm{tr}(P_i) + \mathrm{tr}(Q_i) + \mathrm{tr}(R_i)\right]$ 2

$$= \sum(-1)^i \,\mathrm{tr}(Q_i) = \Lambda(\theta_*) .$$

(c) Lefschetz Fixed Point Theorem.
If X is triangulable & $f: X \to X$ has $\Lambda(f_*) \neq 0$ then f has a fixed point. 2

(ii)(a)

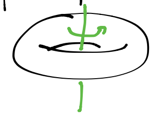


(a) $H_i(M(g)) = \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}^{2g} & i=1 \\ \mathbb{Z} & i=2 \\ 0 & \text{ow} \end{cases}$ 2 If $f \simeq id$

$\Lambda(f) = \Lambda(id) = 1 - 2g + 1 = 2 - 2g$ 3

All individual statements seen – but perhaps not in this combination!

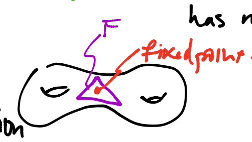
Hence, by LFPT, if $g \neq 1$ there is a fixed point.
For $g = 1$ we may use rotation R_θ



. $t \mapsto R_{\theta t}$ defines a homotopy 2 $id = R_0 \simeq R_\theta$ & R_θ has no fixed pt.

Unseen.

(b) If f has exactly one fixed point we suppose F is the sub complex of simplices it lies in. This is a cone on P & hence contractible. Let $X = M(g) \setminus F$. 2



Now f maps F to F & hence $f|_X$ is a self map of X. & by construction it has no fixed point. 3
However $H_i(X) = H_i\, M(g)$ for $i = 0, 1$. 2 (either use M-V & $M(g) = X \cup F$, or use $X \simeq \bigvee_{i=1}^{2g} S^1$).

So $\Lambda(f|_X) = 1 - 2g \neq 0$ ※ LFPT.)

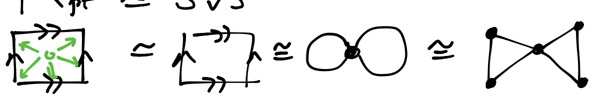
In all cases, 2 for correct answer, 3 with some reason

A5: (i) True. Since $H_*(\mathbb{RP}^2; \mathbb{Q}) = H_*(pt; \mathbb{Q})$, $\Lambda(f) = 1$ for any $f: \mathbb{RP}^2 \to \mathbb{RP}^2$
The statement follows from the Lefschetz Fixed Point Thm [Seen example] 5

(ii) False. If $\mathbb{R}^2 \cong \mathbb{R}^3$ then $S^1 \simeq \mathbb{R}^2 \setminus \{pt\} \cong \mathbb{R}^3 \setminus \{pt\} \simeq S^2$ 5
However $S^1 \not\simeq S^2$, either since $\pi_1(S^1, x) \cong \mathbb{Z} \not\cong 1 = \pi_1(S^2, y)$
or since $H_1(S^1) \cong \mathbb{Z} \not\cong 0 = H_1(S^2)$ [Argument seen]
[or by one pt cpt^n & H_2)

(iii) False. $\pi_1(T, x) \simeq \mathbb{Z} \times \mathbb{Z}$ is not a subgroup of $\pi_1(S^2, y) = 1$. [Unseen] 5

(iv) True. $T^2 \setminus pt \simeq S^1 \vee S^1$ 5



[Homotopy equivalence seen]

(v) False. $H_2 S^2 \cong \mathbb{Z}$
$H_2(X) \cong \mathbb{Z} \times \mathbb{Z}$ these are not isomorphic. [In both cases one can use MV or simply triangulate & (since 2 = dim) find 5

$$H_2 = \ker(C_2 \xrightarrow{\partial} C_1).$$

Take $X = S^2 \cup \overline{B^2}$. Homology of S^1 & S^2 known &
$S^2 \cap \overline{B^2} = S^1_E$ top of M-V sequence reads

$$\cap \longrightarrow \oplus \longrightarrow \cup$$

② $0 \longrightarrow H_2S^2 \oplus 0 \longrightarrow H_2(X)$

① $\hookrightarrow H_1(S^1_E) \to 0 \oplus 0$