SUMMARY OF EXAMPLES FOR MAS61015 (ALGEBRAIC TOPOLOGY)

N. P. STRICKLAND

1. Examples of spaces

Here $X \simeq Y$ means that X and Y are homeomorphic, and $X \cong Y$ means that X and Y are homotopy equivalent.

- $\mathbb{R}^n$ (contractible)
- Balls, cubes and simplices (all contractible)
- Spheres, especially S^1, S^2 and S^3
- The Möbius strip $\cong S^1$
- Surfaces: the torus, the projective plane, the closed orientable surface $M(g)$ of genus g.
- The n-torus $T^n = S^1 \times \cdots \times S^1$.
- One-dimensional spaces, especially bouquets of circles, letters of the alphabet and similar examples. Homeomorphism types of these spaces can often be distinguished using cutting invariants, as discussed in Section 1 of the notes.
- Various complements:

$$S^n \setminus \text{ point } \simeq \mathbb{R}^n$$
$$\mathbb{R}^n \setminus \text{ point } \cong S^{n-1}$$
$$S^n \setminus S^m \simeq \mathbb{R}^n \setminus \mathbb{R}^m \cong S^{n-m-1}$$
$$\mathbb{R}^n \setminus \{a_1, \dots, a_m\}$$
$$\mathbb{C} \setminus \mathbb{Z}$$

- The punctured torus $T \setminus \text{point}$ is homotopy equivalent to the figure eight space $E = S^1 \vee S^1$ (the union of two circles that meet at a single point). More generally, $M(g) \setminus \text{point} \cong \bigvee_{i=1}^{2g} S^1$.
- Spaces of matrices: $GL_n(\mathbb{R})$, $O(n)$ and so on. Especially $SO(2) \simeq U(1) \simeq S^1$ and $SL_2(\mathbb{R}) \simeq \mathbb{R}^2 \times S^1$ and $GL_2(\mathbb{R}) \simeq \{1, -1\} \times \mathbb{R}^3 \times S^1$.
- $\mathbb{R}P^n$ and $\mathbb{C}P^n$ for general n, either as quotients or as spaces of projection matrices. Especially $\mathbb{R}P^1 \simeq S^1$ and $\mathbb{R}P^2$.
- Most of the above spaces are subspaces of $\mathbb{R}^N$ for some N. This means that they are Hausdorff, and are compact iff they are bounded and closed.
- Balls, cubes, simplices, spheres, tori and closed surfaces are compact. The spaces $\mathbb{R}^n$ and $S^n \setminus \text{point}$ and $S^n \setminus S^k$ are not compact.
- $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are compact and Hausdorff.
- Examples of non-Hausdorff spaces include the doubled line, the Sierpiński space and any indiscrete space with at least two points.

2. Homology

- The following spaces are contractible and so have $H_*(X) = \mathbb{Z}$: $\mathbb{R}^n$, balls, cubes, simplices.
- If $f\colon B^k \to S^n$ is continuous and injective then again $H_*(S^n \setminus f(B^k)) = \mathbb{Z}$ (even though $S^n \setminus f(B^k)$ is not always contractible).
- For $n > 0$ we have $H_0(S^n) = H_n(S^n) = \mathbb{Z}$, and $H_i(S^n) = 0$ for $i \neq 0, n$. We have mentioned several spaces that are homeomorphic or homotopy equivalent to S^n for some n, and so have the same homology: the Möbius strip, $\mathbb{R}P^1$, $SO(2)$ and $U(1)$ are all equivalent to S^1, and $GL_2(\mathbb{C}) \cong SU(2) \simeq S^3$, and $\mathbb{R}^n \setminus 0 \cong S^{n-1}$, and $S^{n+k+1} \setminus S^k \simeq \mathbb{R}^{n+k+1} \setminus \mathbb{R}^k \cong S^n$.

- If $f\colon S^k \to S^n$ is continuous and injective then $H_*(S^n \setminus f(S^k)) \simeq H_*(S^{n-k-1})$ (even though $S^n \setminus f(S^k)$ is not always homotopy equivalent to S^{n-k-1}). For the case $n = k$, one should interpret S^{n-k-1} as $\emptyset$.
- For $X = \mathbb{C}P^n$ we have $H_0(X) = H_2(X) = \cdots = H_{2n}(X) = \mathbb{Z}$, and all other homology groups are trivial.
- The homology groups of $\mathbb{R}P^1 \simeq S^1$ are $(\mathbb{Z}, \mathbb{Z})$. The homology groups of $\mathbb{R}P^2$ are $(\mathbb{Z}, \mathbb{Z}/2)$. More generally, for $m \geq 0$ we have

$$H_k(\mathbb{R}P^{2m+1}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \text{ or } k = 2m+1 \\ \mathbb{Z}/2 & \text{if } 0 < k < 2m \text{ and } k \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

$$H_k(\mathbb{R}P^{2m+2}) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/2 & \text{if } 0 < k < 2m+2 \text{ and } k \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

  We also have

$$H_k(\mathbb{R}P^n; \mathbb{Z}/2) = \begin{cases} \mathbb{Z}/2 & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise.} \end{cases}$$

- If X is a connected one-dimensional space consisting of some vertices and edges between them, then $H_*(X) \simeq (\mathbb{Z}, \mathbb{Z}^m)$ for some m.
- The torus $T^2 = S^1 \times S^1$ has $H_*(T^2) = (\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z})$. More generally, we have $H_k(T^n) = \mathbb{Z}^{\binom{n}{k}}$ for $0 \leq k \leq n$.
- If X is a closed orientable surface of genus g, then $H_*(X) = (\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z})$.
- For any X, we have $H_0(X) = \mathbb{Z}\{\pi_0(X)\}$. Thus, if X has n path components, then $H_0(X) = \mathbb{Z}^n$. In particular, if X is path-connected then $H_0(X) = \mathbb{Z}$.
- If X is a disjoint union $Y \amalg Z$ then $H_*(X) = H_*(Y) \oplus H_*(Z)$.
- Suppose that $X = U \cup V$, where U and V are open and $U \cap V$ is contractible. Then $H_i(X) = H_i(U) \oplus H_i(V)$ when $i > 0$, but there is a small adjustment in degree zero. If U has n path-components and V has m path-components then $H_0(X) = \mathbb{Z}^{n+m-1}$ (whereas $H_0(U) \oplus H_0(V) = \mathbb{Z}^{n+m}$). In particular, if U and V are both path-connected then so is X, and in this case we have $n = m = n + m - 1 = 1$ so $H_0(X) \simeq H_0(U) \simeq H_0(V) \simeq \mathbb{Z}$.