EXAMINABLE CONTENT FOR MAS61015 (ALGEBRAIC TOPOLOGY)

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This is the long version of this document, listing all definitions and results in the notes, whether or not they are starred. There is also a shorter version that lists only the starred points.

For starred definitions, you may be required to state the definition in the exam. For singly starred results, you may be required to state the result in the exam. For doubly starred results, you may be asked to reproduce the proof in the exam. You may also be asked to prove something where the required argument is a step in one of these specified proofs.

Although you will not be examined directly on the unstarred points, understanding those points may nonetheless be helpful for solving problems posed in the exam.

- Letters of the alphabet (Example 1.1)
- Cut point (Definition 1.2)
- * Simplex (Definition 1.3)
- Miscellaneous spaces (Example 1.4)
- Metric, metric space (Definition 3.1)
- Metrics d_1, d_2, d_∞ on $\mathbb{R}^n$ (Example 3.2)
- The trace metric on $M_n(\mathbb{R})$ (Example 3.3)
- Continuous map (metric version) (Definition 3.4)
- Distance reducing maps are continuous (Lemma 3.5)
- Open ball (Definition 3.6)
- Open set (metric version) (Definition 3.7)
- Different definitions of metric closure (Remark 3.8)
- Properties of open sets in a metric space (Proposition 3.9)
- Preimage (Definition 3.10)
- Properties of preimages (Lemma 3.11)
- Conditions for metric continuity (Proposition 3.12)
- * Topology, topological space (Definition 3.13)
- * The metric topology (Example 3.14)
- * The discrete and indiscrete topologies (Example 3.15)
- Equivalent metrics give the same topology (Lemma 3.16)
- d_1, d_2 and d_∞ give the same topology on $\mathbb{R}^n$ (Corollary 3.17)
- Nested balls for different norms on $\mathbb{R}^n$ (Remark 3.18)
- * Continuous map (Definition 3.19)
- Continuity is the same as metric continuity (Remark 3.20)
- Building up continuous functions (Example 3.21)
- Constant functions are continuous (Example 3.22)
- Continuity in terms of closed sets (Proposition 3.23)
- ** Composites of continuous maps are continuous (Proposition 3.24)
- * Subspace topology (Definition 3.25)
- ** The subspace topology is a topology (Proposition 3.26)
- Inclusion maps are continuous (Lemma 3.27)
- Restriction and corestriction of continuous maps (Proposition 3.28)
- ** Maps into subspaces with the subspace topology (Proposition 3.29)
- Open subsets of open subsets (Lemma 3.30)
- Closed sets in the subspace topology (Lemma 3.31)
- Closed subsets of closed sets (Lemma 3.32)
- Openness in terms of neighbourhoods (Lemma 3.33)

- Open patching (Proposition 3.34)
** Closed patching (Proposition 3.35)
* Homeomorphism, homeomorphic (Definition 4.1)
- Homeomorphism vs homotopy equivalence (Remark 4.2)
- Few indirect methods for homeomorphism (Remark 4.3)
- Homeomorphisms of finite intervals (Proposition 4.4)
- Homeomorphisms of infinite intervals (Proposition 4.5)
- Checking function definitions (Remark 4.6)
- Geometry for infinite interval homeomorphisms (Remark 4.7)
** An open ball is homeomorphic to euclidean space (Example 4.8)
** $\mathbb{R}^n \setminus \{0\}$ is homeomorphic to $(0, \infty) \times S^{n-1}$ (Example 4.9)
** A continuous bijection that is not a homeomorphism (Example 4.10)
- Topology of the matrix spaces O_2 and $GL_2(\mathbb{R})$ (Example 4.11)
- Avoid polar coordinates (Remark 4.12)
- Stereographic projection (Example 4.13)
- Path (Definition 5.1)
- An example of a path in an annulus (Example 5.2)
- Paths with the same image can be different (Remark 5.3)
- Straight line paths (Example 5.4)
- Identifying Δ_1 with $[0, 1]$ (Remark 5.5)
* Join, reversal and constant paths (Definition 5.6)
** Validating path operations (Remark 5.7)
* Relation of points being connected (Definition 5.8)
** The relation based on paths is an equivalence relation (Proposition 5.9)
* The set $\pi_0(X)$ (Definition 5.10)
** Convex sets are path connected (Example 5.11)
- An example of $\pi_0(Y)$ (Example 5.12)
- $\pi_0(Y)$ as a set of sets (Remark 5.13)
** Spheres are path connected (Proposition 5.14)
** Disconnection results from the Intermediate Value Theorem (Proposition 5.15)
- $\pi_0(\mathbb{Z})$ can be identified with $\mathbb{Z}$ (Example 5.16)
- $GL_2(\mathbb{R})$ has two path components (Example 5.17)
- Disconnection results from disjoint open sets (Proposition 5.18)
- Continuous maps preserve the path relation (Lemma 5.19)
- Functorial properties of $\pi_0(X)$ (Proposition 5.20)
- Category (Definition 6.1)
- The category of groups (Example 6.2)
- Concrete and non-concrete categories (Remark 6.3)
- Various algebraic categories (Example 6.4)
- Two categories of metric spaces (Example 6.5)
- The category Top (Example 6.6)
* Functor (Definition 6.7)
- π_0 as a functor (Example 6.8)
- The functor from metric spaces to toplogical spaces (Example 6.9)
- The functor $(-) \times \mathbb{Z}$ (Example 6.10)
- The functors $A \mapsto 2A$ and $A \mapsto A/2A$ (Example 6.11)
* Inverse, isomorphism (Definition 6.12)
** Inverses are unique (Lemma 6.13)
- Isomorphisms in various specific categories (Example 6.14)
- Properties of isomorphisms in general categories (Proposition 6.15)
- Isomorphism as an equivalence relation (Corollary 6.16)
** Functors preserve isomorphisms (Proposition 6.17)
- Functors preserve isomorphic objects (Corollary 6.18)

- Retract, retraction pair (Definition 6.19)
- G is a retract of $G \times H$ (Example 6.20)
- S^2 is a retract of $\mathbb{R}^3 \setminus \{0\}$ (Example 6.21)
- Retracts of finite sets (Example 6.22)
- Injections, surjections and retraction pairs (Proposition 6.23)
- Non-retraction examples with groups (Corollary 6.24)
- Functors preserve retracts and reflect non-retracts (Proposition 6.25)
- Non-retraction results using π_0 (Proposition 6.26)
- Coproduct topology (Definition 7.1)
- Coproduct and subspace topologies are compatible (Remark 7.2)

** Maps from a coproduct to another space (Proposition 7.3)
- Coproducts in general categories (Remark 7.4)
- Maps to a coproduct from another space (Proposition 7.5)

* Product topology, boxes (Definition 7.6)
- Illustration of the product topology (Example 7.7)
- Boxes are open (Lemma 7.8)

** The product topology is a topology (Proposition 7.9)
- Product projections are continuous (Lemma 7.10)

** Maps to a product from another space (Proposition 7.11)
- Products in general categories (Remark 7.12)
- Equivalence relation (Definition 7.13)
- Function saturated for an equivalence relation (Definition 7.14)
- Maps from a quotient set to another set (Proposition 7.15)
- Maps from a quotient set to another set (Corollary 7.16)

* Quotient topology (Definition 7.17)

** The quotient topology is a topology (Proposition 7.18)

** Maps from a quotient space to another space (Proposition 7.19)
- Maps from a quotient space to another space (Corollary 7.20)
- Coequalisers in general categories (Remark 7.21)
- Projective spaces as quotient spaces (Example 7.22)
- $\mathbb{R}P^1$ is homeomorphic to S^1 (Example 7.23)
- Two discs glued along the boundary give a sphere (Example 7.24)
- An orientable surface as a quotient of an octagon (Example 7.25)
- The torus as $\mathbb{R}^2/\mathbb{Z}^2$ (Example 7.26)
- The doubled line (Example 7.27)
- We have no good topology for mapping spaces (Remark 7.28)
- π_0 preserves coproducts (Proposition 7.29)

** π_0 preserves products (Proposition 7.30)

* Hausdorff separation, Hausdorff space (Definition 8.1)

** Metric spaces are Hausdorff (Example 8.2)
- Nontrivial indiscrete spaces are not Hausdorff (Example 8.3)
- The Sierpiński space is not Hausdorff (Example 8.4)
- The doubled line is not Hausdorff (Example 8.5)

** Subspaces of Hausdorff spaces are Hausdorff (Proposition 8.6)
- Coproducts of Hausdorff spaces are Hausdorff (Proposition 8.7)

** Products of Hausdorff spaces are Hausdorff (Proposition 8.8)

* Compact space, open cover, finite subcover (Definition 8.9)
- $\mathbb{R}$ is not compact (Example 8.10)
- $(0, 1)$ is not compact (Example 8.11)
- Bounded set (Definition 8.12)
- Finite spaces are compact (Lemma 8.13)
- In $\mathbb{R}^n$, compact means bounded and closed (Proposition 8.14)
- Various compact subspaces of $\mathbb{R}^n$ (Example 8.15)

- The orthogonal group O_n is compact (Example 8.16)
** Closed subspaces of compact spaces are compact (Proposition 8.17)
- Compact subspaces of Hausdorff spaces are compact (Proposition 8.18)
- In a compact Hausdorff space, compact means closed (Corollary 8.19)
- Images of compact spaces are compact (Proposition 8.20)
- Quotients of compact spaces are compact (Corollary 8.21)
- Projective spaces are compact (Example 8.22)
- Coproducts of compact spaces are compact (Proposition 8.23)
- Products of compact spaces are compact (Theorem 8.24)
- Finitely covered subset (Definition 8.25)
- Finitely covered strips (Lemma 8.26)
- A continuous bijection from compact to Hausdorff is a homeomorphism (Proposition 8.27)
- Gluing two discs, revisited (Example 8.28)
- The torus as $\mathbb{R}^2/\mathbb{Z}^2$, revisited (Example 8.29)
* Lebesgue number (Definition 8.30)
- Lebesgue numbers exist (Proposition 8.31)
- Bogus definition of homotopy (Definition 9.1)
* Homotopy (Definition 9.2)
** Being homotopic is an equivalence relation (Proposition 9.3)
- Homotopy class, set of homotopy classes (Definition 9.4)
- Winding numbers (Example 9.5)
- Straight line homotopies (Example 9.6)
- Homotopy is compatible with composition (Proposition 9.7)
- Well-defined composition of homotopy classes (Corollary 9.8)
* The category hTop (Definition 9.9)
* Homotopy equivalence (Definition 9.10)
- Homeomorphisms are homotopy equivalences (Lemma 9.11)
- S^{n-1} is homotopy equivalent to $\mathbb{R}^n \setminus \{0\}$ (Proposition 9.12)
- Contraction (Definition 9.13)
** Convex sets are contractible (Example 9.14)
- S^2 is not contractible (Example 9.15)
- The one-point space (Definition 9.16)
- Contractible means homotopy equivalent to 1 (Proposition 9.17)
- $[0,1] \to 1$ is a homotopy equivalence but not a homeomorphism (Remark 9.18)
- Being homotopy equivalent is an equivalence relation (Proposition 9.19)
- Products preserve homotopy equivalences (Proposition 9.20)
- Various homotopy equivalences related to circles (Example 9.21)
* Homotopy retract, homotopy retraction (Definition 9.22)
- $\mathbb{R}^2 \setminus \{0\}$ is a homotopy retract of the figure 8 (Example 9.23)
** Homotopy retracts of contractible spaces are contractible (Proposition 9.24)
** π_0 is homotopy invariant (Proposition 9.25)
* The set $S_k(X)$ of singular k-simplices (Definition 10.1)
- $S_0(X)$ is the set of points in X (Example 10.2)
- $S_1(X)$ is the set of paths in X (Example 10.3)
* Linear simplices (Example 10.4)
- An example with several singular simplices (Example 10.5)
- Free abelian group generated by a set (Definition 10.6)
- If $|P| = n < \infty$ then $\mathbb{Z}\{P\} \simeq \mathbb{Z}^n$ (Remark 10.7)
* The group $C_k(X)$ of singular k-chains (Definition 10.8)
- An example with several singular chains (Example 10.9)
- Path join is not the same as sum of 1-chains (Remark 10.10)
- Preliminary version of the algebraic boundary operator (Predefinition 10.11)
* The maps $\delta_i \colon \Delta_{n-1} \to \Delta_n$ (Definition 10.12)

- Examples of $\delta_i\colon \Delta_{n-1} \to \Delta_n$ (Example 10.13)
* The algebraic boundary map $\partial\colon C_n(X) \to C_{n-1}(X)$ (Definition 10.14)
- Examples of the algebraic boundary map (Example 10.15)
- $\partial^2 = 0\colon C_k(X) \to C_{k-2}(X)$ (Proposition 10.16)
- Examples illustrating $\partial^2 = 0$ (Example 10.17)
- If $i < j$ then $\delta_j\delta_i = \delta_i\delta_{j-1}$ (Lemma 10.18)
- Example of $\delta_j\delta_i = \delta_i\delta_{j-1}$ (Example 10.19)
** $\partial^2 = 0\colon C_4(X) \to C_2(X)$ (Example 10.20)
* The homology group $H_k(X)$ (Definition 10.21)
- $[z]$ is only meaningful when $\partial(z) = 0$ (Remark 10.22)
** Homology of a point (Proposition 10.23)
- Homology of a discrete space (Remark 10.24)
- $H_0(X) \simeq \mathbb{Z}\{\pi_0(X)\}$ (Proposition 10.25)
- Example of $H_0(X) \simeq \mathbb{Z}\{\pi_0(X)\}$ (Example 10.26)
- Interaction of path operations and homology (Lemma 10.27)
- Loop (Definition 10.28)
- $H_1(X)$ in terms of loops (Proposition 10.29)
- Filling in of a loop (Definition 10.30)
- Fillable loops give zero in homology (Lemma 10.31)
- The principal logarithm (Definition 11.1)
- The set $\mathrm{LOG}(z)$ of all logarithms (Definition 11.2)
- Log lifting of paths (Lemma 11.3)
- Log lifting with convex domain (Corollary 11.4)
- The total angle map $\omega\colon C_1(\mathbb{C}^\times) \to \mathbb{C}$ (Definition 11.5)
- Winding number of the standard loop u_n (Example 11.6)
- Boundaries have total angle zero (Lemma 11.7)
- The maps $\beta\colon \mathbb{C} \to \mathbb{C}^\times$ and $\gamma\colon C_0(\mathbb{C}^\times) \to \mathbb{C}^\times$ (Definition 11.8)
- $\beta(\omega(u)) = \gamma(\partial(u))$ (Lemma 11.9)
- Winding numbers for homology classes (Corollary 11.10)
- First homology of a circle (Theorem 11.11)
* Homomorphisms between cyclic groups (Lemma 12.1)
* The Chinese Remainder Theorem (Proposition 12.2)
* Primary decomposition of cyclic groups (Corollary 12.3)
* Finitely generated abelian group (Definition 12.4)
- Examples of finite generation (Example 12.5)
- $\mathbb{Z}[x]$ is not finitely generated (Example 12.6)
** $\mathbb{Q}$ is not finitely generated (Example 12.7)
- Subgroups of finitely generated abelian groups are finitely generated (Proposition 12.8)
- Classification of finitely generated abelian groups (Theorem 12.9)
- Example of classifying a group (Example 12.10)
- Classifying abelian groups of order 72 (Example 12.11)
* Exact sequence (Definition 12.12)
- Unfolding the definition of short exact sequences (Remark 12.13)
- Various examples of exact sequences (Example 12.14)
** Various properties of exact sequences (Proposition 12.15)
- Long exact sequence (Definition 12.16)
- Adding zeros to short exact sequences (Example 12.17)
- Doubling on $\mathbb{Z}/4$ is exact (Example 12.18)
- Isomorphism from exactness (Lemma 12.19)
** Orders in short exact sequences (Lemma 12.20)
* Chain complex, cycles, boundaries, homology (Definition 13.1)
- The notation A_* (Remark 13.2)
- Singular chains as a chain complex (Remark 13.3)

- Exactness means vanishing homology (Remark 13.4)
- Chain complexes with only one group (Example 13.5)
- Chain complexes with only two groups (Example 13.6)
- Chain complex of an n-gon (Example 13.7)
- Method for computing homology (Remark 13.8)

* Chain map between chain complexes (Definition 13.9)
- Composition of chain maps (Definition 13.10)

** Z_*, B_* and H_* as functors of chain complexes (Proposition 13.11)

* Homology as a functor on spaces (Construction 13.12)
- Homology, homeomorphism and retracts (Remark 13.13)

** Homological effect of constant maps (Remark 13.14)

* Chain homotopy (Definition 14.1)
- First geometric picture of chain homotopy (Remark 14.2)
- Chain homotopy for twisting an n-gon (Example 14.3)
- Being chain homotopic is an equivalence relation (Proposition 14.4)
- Chain homotopy is compatible with composition (Proposition 14.5)
- The category hChain (Definition 14.6)

** Chain homotopic chain maps have the same effect in homology (Proposition 14.7)
- Homotopic continuous maps have the same effect in homology (Proposition 14.8)
- The maps $\zeta_i\colon \Delta_{k+1} \to [0,1] \times \Delta_k$ (Definition 14.9)
- Examples of $\zeta_i\colon \Delta_{k+1} \to [0,1] \times \Delta_k$ (Example 14.10)
- The chain homotopy $\sigma_k\colon C_k(X) \to C_{k+1}(Y)$ (Definition 14.11)
- Faces of prisms (Lemma 14.12)
- Prisms give a chain homotopy (Proposition 14.13)

** Homotopy equivalences give isomorphisms in homology (Corollary 14.14)
- Homology as a functor from hTop (Remark 14.15)
- Homological effect of maps homotopic to a constant (Proposition 14.16)
- Homology of contractible spaces (Proposition 14.17)

* The Mayer-Vietoris sequence (Theorem 15.1)
- Truncating the Mayer-Vietoris sequence (Proposition 15.2)

** Homology of spheres (Theorem 15.3)
- Degree of homology classes (Remark 15.4)

** Different spheres are not homotopy equivalent (Theorem 16.1)

** Different euclidean spaces are not homeomorphic (Theorem 16.2)

** The Brouwer theorem: any $f\colon B^n \to B^n$ has a fixed point (Theorem 16.3)
- The Brouwer map $m\colon X_n \to S^{n-1}$ (Definition 16.4)
- Properties of the Brouwer map $m\colon X_n \to S^{n-1}$ (Lemma 16.5)
- The Fundamental Theorem of Algebra: roots of complex polynomials (Theorem 16.6)
- Mayer-Vietoris exactness I (Proposition 17.1)
- Long exact sequence from a short exact sequence (Theorem 17.2)

* Snakes (Definition 17.3)
- Existence of snakes (Lemma 17.4)
- Uniqueness of snakes (Lemma 17.5)

* The connecting map δ for the Snake Lemma (Definition 17.6)
- The connecting map is a homomorphism (Remark 17.7)
- The Snake Lemma movie (Remark 17.9)
- Mayer-Vietoris exactness II (Proposition 17.10)

** Mayer-Vietoris exactness III (Proposition 17.11)
- The cone map $\beta\colon C_k^{\mathrm{lin}}(\Delta_n) \to C_{k+1}^{\mathrm{lin}}(\Delta_n)$ (Definition 18.1)
- Cones for nonlinear simplices (Remark 18.2)
- Key property of the cone map (Lemma 18.3)
- Definition of subdivision using θ_n and θ'_n (Definition 18.4)
- Subdivision of 2-simplices (Example 18.5)

- Subdivision of 1-simplices (Example 18.6)
- Subdivision in terms of permutations (Remark 18.7)
- Naturality of subdivision (Lemma 18.8)
- $\theta_n = \mathrm{sd}(\iota_n)$ and $\theta'_n = \mathrm{sd}(\partial(\iota_n))$ (Lemma 18.9)
- Subdivision is a chain map (Proposition 18.10)
- Subdivision is chain homotopic to the identity (Proposition 18.11)
- Diameter of a chain (Definition 18.12)
- Diameter of a subdivided chain (Lemma 18.13)
- Subgroups used for Mayer-Vietoris (Definition 19.1)
- Some subgroups are not subcomplexes (Remark 19.2)
- Iterated subdivision makes chains small (Lemma 19.3)
- The quotient by small chains is acyclic (Corollary 19.4)
- Small chains have the same homology (Corollary 19.5)
- Homology of unions with contractible intersection (Lemma 20.1)

** Homology of convex open set with removed points (Proposition 20.2)
- Homology of a ladder (Example 20.3)
- Homology of a bouquet of circles (Example 20.4)

** Homology of the torus (Proposition 20.5)
- Homology of the torus, alternative method (Remark 20.6)

** Homology of the projective plane (Projective 20.7)
- Homology of closed surfaces (Remark 20.8)
- Some homology of projective spaces (Proposition 20.9)
- Full homology of projective spaces (Remark 20.10)

* The Generalised Jordan Curve Theorem (Theorem 21.1)
- Basic case of the Generalised Jordan Curve Theorem (Remark 21.2)
- Codimension one spheres separate $\mathbb{R}^n$ (Theorem 21.3)
- Acyclic space (Definition 21.4)
- Acyclic vs contractible (Remark 21.5)
- Acyclicity and the map to a point (Lemma 21.6)
- A sphere minus a ball is acyclic (Theorem 21.7)
- Acyclic complements of unions (Lemma 21.8)
- Slices and coslices (Definition 21.9)
- Acyclic complements and fine slices (Remark 21.10)
- Detecting nontrivial homology (Lemma 21.11)
- Chains in an infinite union (Lemma 21.12)
- Homology of an infinite union (Corollary 21.13)
- Euclidean space is locally connected (Lemma 21.14)
- Images of open balls are open (Proposition 21.15)
- Invariance of Domain (Corollary 21.16)
- Fibres of the exponential map (Example 22.1)

* Covering map, trivially covered set (Definition 22.2)
- Subsets of trivially covered sets are trivially covered (Remark 22.3)
- Product projections as coverings (Example 22.4)
- Squaring complex numbers gives a covering (Example 22.5)
- The defining covering of $\mathbb{R}P^n$ (Proposition 22.6)
- The exponential map is a covering (Proposition 22.7)
- Lifts of points, paths and maps (Definition 22.8)
- Unique lifts of small maps (Lemma 22.9)
- Path lifting (Proposition 22.10)
- The endpoint map (Remark 22.11)
- Overlapping lifts are identical (Corollary 22.12)
- Lifting maps with convex domain (Proposition 22.13)
- n-sheeted covering (Definition 23.1)

- Sheets of the nth power map (Example 23.2)
- Sheets of a product projection (Example 23.3)
- Sheets for projective space (Example 23.4)
- Lifts of a singular simplex (Lemma 23.5)

* The transfer map (Definition 23.6)

- Example of a 3-sheeted covering (Example 23.7)

** The transfer is a chain map (Proposition 23.8)

- The transfer in homology (Remark 23.9)
- Homology mod 2 (Definition 23.10)
- Modular homology when homology is free (Remark 23.11)
- Reduced modular chains (Remark 23.12)
- Properties of modular homology (Remark 23.13)
- A short exact sequence related to the transfer (Lemma 23.14)
- Modular homology of $\mathbb{R}P^n$ (Theorem 23.15)
- Antipodal maps (Definition 24.1)
- The inclusion is antipodal (Example 24.2)
- Antipodal maps induce maps of projective spaces (Remark 24.3)
- Antipodal maps give the identity in modular homology (Proposition 24.4)

* The Borsuk-Ulam theorem: no antipodal maps $S^{>m} \to S^m$ (Theorem 24.5)

- Points of evenness exist (Corollary 24.6)
- Antipodal example with temperature and pressure (Example 24.7)
- The Sandwich Slicing Theorem (Theorem 24.8)
- Sandwich explanation (Example 24.9)
- Slicing spherical sandwiches (Example 24.10)