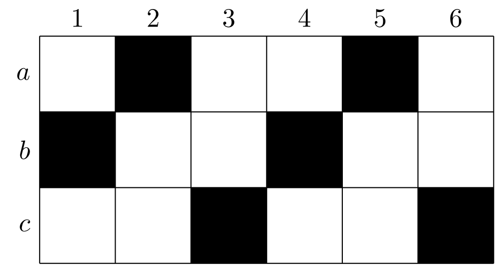
MAS334 COMBINATORICS — PROBLEM SHEET 5 — Solutions

Please hand in exercises 5.1 and 5.7 by the end of Week 11.

Exercise 5.1. Consider the following team allocation problem, in which each job needs a team of two people.



(a) Find $|C_U|$ for $U \subseteq \{a,b,c\}$, and thus check that the team allocation problem is plausible.
(b) Find an explicit solution.

Solution:

(a) The team allocation plausibility is that $|C_U| \geq m_U$ for all $U \subseteq B = \{a,b,c\}$. Here m_U is the total team size needed for all the jobs in U. As each job needs two people, this is just $m_U = 2|U|$. Thus, the plausibility condition is that $|C_U| \geq 2|U|$ for all $U \subseteq B = \{a,b,c\}$. For $U = \emptyset$ we have $C_U = \emptyset$ and so $|C_U| = |U| = 0$ and the condition is satisfied. For sets of size one, we have

$$C_a = \{1,3,4,6\} \qquad C_b = \{2,3,5,6\} \qquad C_c = \{1,2,4,5\}.$$

Thus, whenever $|U| = 1$ we have $|C_U| = 4$ so the condition is again satisfied. From the above we also see that

$$C_a \cup C_b = C_a \cup C_c = C_b \cup C_c = \{1,2,3,4,5,6\} = A,$$

so whenever $|U| = 2$ we have $|C_U| = 6$ so the condition is again satisfied. Finally, the only case with $|U| = 3$ is $U = B = \{a,b,c\}$ and here C_U is again equal to A so $|C_U| = 6 = 2|U|$ as required. Thus, every subset of B is plausible, so the team version of Hall's Theorem tells us that there exists a solution.

(b) In fact, it is not hard to find a solution by inspection: we can allocate people 1 and 2 to job c, and people 3 and 4 to job a, and people 5 and 6 to job b.

Exercise 5.2. Which of the following are possible scores in a tournament of 8 people?

(i) $6,6,5,5,3,2,1,1$.
(ii) $6,6,6,5,2,2,1,0$.
(iii) $6,6,5,5,3,2,1,0$.

Solution: Landau's Theorem says that a list $s_1, \dots, s_8$ numbers can be the score list of an 8-person tournament iff

(a) $s_1 + \dots + s_8 = \binom{8}{2} = 28$; and
(b) The sum of any k of the s_i's is at least $\binom{k}{2}$.

If $s_1 \geq s_2 \geq \cdots \geq s_8$, then condition (b) is equivalent to the condition that the sum of the last k of the s_i must be at least $\binom{k}{2}$.

For list (i), the sum of the entries is 29, so this cannot be the score list from a tournament.

For list (ii), the sum of the last 4 entries is 5, which is less than $\binom{5}{2}$, so this cannot be the list of scores from a tournament. (A reminder of the reason: the last 4 players play $\binom{4}{2} = 6$ games against each other, so they must earn a total score of 6 from those games, even if they all lose against everyone else.)

Now consider list (iii):

$$s_8 = 0 \geq 0 = \binom{1}{2} \qquad s_7 + s_8 = 1 \geq 1 = \binom{2}{2}$$

$$s_6 + s_7 + s_8 = 3 \geq 3 = \binom{3}{2} \qquad s_5 + s_6 + s_7 + s_8 = 6 \geq 6 = \binom{4}{2}$$

$$s_4 + \cdots + s_8 = 11 \geq 10 = \binom{5}{2} \qquad s_3 + \cdots + s_8 = 16 \geq 15 = \binom{6}{2}$$

$$s_2 + \cdots + s_8 = 22 \geq 21 = \binom{7}{2} \qquad s_1 + \cdots + s_8 = 28 = \binom{8}{2}.$$

Landau's Theorem tells us that there exists a tournament with these scores. For the simplest example of such a tournament, suppose that the lower-numbered player wins every game except that player 4 beats player 1. Then the results and scores are as follows:

	1	2	3	4	5	6	7	8	score
1		W	W	L	W	W	W	W	6
2	L		W	W	W	W	W	W	6
3	L	L		W	W	W	W	W	5
4	W	L	L		W	W	W	W	5
5	L	L	L	L		W	W	W	3
6	L	L	L	L	L		W	W	2
7	L	L	L	L	L	L		W	1
8	L	L	L	L	L	L	L		0

Exercise 5.3. In a tournament of n players, let the score of player i be w_i. Let l_i denote the number of games lost by player i.

(a) Give a formula for l_i (in terms of n and w_i).
(b) Show that $l_1, \ldots, l_n$ are the scores of a tournament.

Solution:

(a) Each player plays $n-1$ games, one against each of the other players. Each game results in a win for one of the players. So if player i wins w_i games, they lose $n-1-w_i$ games. So $l_i = n-1-w_i$.

(b) Clearly there is a tournament in which the result of every game is the opposite to that in the given tournament. The scores of this tournament are $l_1, \ldots, l_n$.

Alternatively, use Landau's Theorem:

Since $w_1, \ldots, w_n$ are the scores of a tournament, by Landau's theorem, any r of them, say $w_{i_1}, \ldots, w_{i_r}$, add to at least $\binom{r}{2}$. Then

$$\begin{aligned}\sum_{j=1}^{r} l_{i_j} &= \sum_{j=1}^{r} n-1-w_{i_j} = r(n-1) - \sum_{j=1}^{r} w_{i_j} \leq r(n-1) - \binom{r}{2} \\ &= rn - r - \binom{r}{2} = rn - \binom{r+1}{2} \\ &= (n-1) + (n-2) + \cdots + (n-r).\end{aligned}$$

Thus any r of the l_is add to at most $(n-1)+(n-2)+\cdots+(n-r)$ and, by Landau's Theorem, these are the scores of a tournament. (This argument is correct, but much less satisfactory than the first one.)

Exercise 5.4. By a *trio* we mean a tournament of three players. If we choose three players from a larger tournament and just consider the games that they play against each other, that gives a trio. A *clear winner* in a trio is a player who beats both the other players. A *clear trio* is a trio that has a clear winner. A *cyclic trio* is a trio in which the players can be labelled a, b and c such that a beats b and b beats c and c beats a.

(a) Show that every trio is either clear or cyclic. What are the score sequences for these two cases?

(b) In a tournament of n players, let w_i be the score of player i. Show that the number of trios in which player i is the clear winner is $\binom{w_i}{2}$.

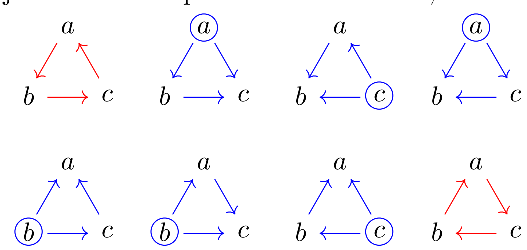
(c) Deduce that the number of cyclic trios is

$$\binom{n}{3} - \sum_{k=1}^{n} \binom{w_k}{2}.$$

Solution:

(a) There are several ways to prove this. One way is to recall (from Proposition 13.9) that there is a winning line, say abc, so a beats b and b beats c. If a beats c then a is the clear winner, so the trio is clear; otherwise c must beat a and the trio is cyclic.

Another approach is to just list all the possible tournaments, as follows:



(The top half has all the possibilities where a beats b, the left half has all the possibilities where b beats c, and we alternate between possibilities where a beats c and where c beats a.) The first and last possibilities are cyclic, and the others have a clear winner, which is circled.

In a clear trio, the clear winner scores 2, the winner of the game between the other two players scores 1, and the loser scores 0, so the score sequence is $2, 1, 0$. In a cyclic trio, each player wins one game, so the score sequence is $1, 1, 1$.

(b) Let W_i be the set of players who are beaten by player i, so $|W_i| = w_i$. Let C_i be the set of trios in which player i is the clear winner. To produce such a trio, we take player i together with a pair of players taken from W_i; so $|C_i| = \binom{w_i}{2}$.

(c) We need to find the number of trios with no clear winner. The total number of trios is $\binom{n}{3}$. The sets C_i are clearly disjoint, so the total number of trios that have a clear winner is just $\sum_i |C_i| = \sum_i \binom{w_i}{2}$. Thus, the total number of trios with no clear winner is $\binom{n}{3} - \sum_i \binom{w_i}{2}$.

Exercise 5.5. This question concerns tournaments of n players $\{p_1, p_2, \ldots, p_n\}$.

(a) How many different sets of scores are there of the games so that p_1's final score is greater than p_2's which is greater than p_3's $\ldots$ which is greater than p_n's? When you have listed their scores also work out the result of each game.

(b) How many different sets of results are there of the games so that two of the players have the same score but, apart from that, all the scores are different?

(c) You are given that p_1 has the highest score and that all the other players tie second. Show that p_1 must win all their games and that n must be even.

Solution:

(a) We are looking for tournaments in which the score list $s_1, \ldots, s_n$ is strictly decreasing. This means that we have n scores that are all different and all in the set $\{0, \ldots, n-1\}$. It is clear that the only possibility is $s_1 = n-1,\ s_2 = n-2,\ \ldots, s_{n-1} = 1,\ s_n = 0$, or more briefly $s_i = n - i$. We also claim that this pattern of scores can only occur if the lower-numbered player wins in every game (so we have a consistent tournament, as in Example 13.5). Indeed, p_1 has a score of $n-1$ and so must beat all of $p_2, \ldots, p_n$. Player p_2 wins $n-2$ games but loses to p_1 and so must beat all of $p_3, \ldots, p_n$. Player p_3 wins $n-3$ games but loses to p_1 and p_2 so must beat $p_4, \ldots, p_n$, and so on.

(b) Now imagine a tournament in which one score occurs precisely twice, and all the other scores are different. We will show that in fact this cannot happen. If it does happen, then there are precisely $n-1$ different scores altogether, and they all lie in the set $\{0, \ldots, n-1\}$, so precisely one element of that set must be missing from the score list. Let i be the score that is missing, and let j be the score that is repeated. We have thus taken the score list from (a) but replaced i by j, so the total of the scores changes by $j-i$. However, the total of the scores must be $\binom{n}{2}$ in both cases, so $j-i=0$, so $j=i$. But this is impossible, because i is missing from the score list and j is not. This contradiction shows that there can be no tournament of the type under consideration.

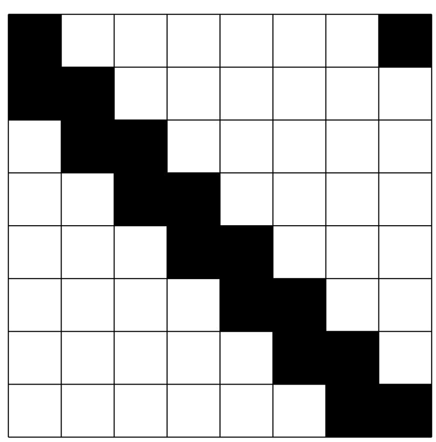
(c) Now consider a tournament where $s_1 = x$ and $s_2 = s_3 = \cdots = s_n = y$ with $x > y$. By considering the sum of all the scores, we get $x + (n-1)y = \binom{n}{2} = n(n-1)/2$, which simplifies to $x = (n-1)(n/2-y)$. By substituting this into the inequality $x > y$ and rearranging, we get $2y < n-1$. On the other hand, Landau tells us that the sum of the last $n-1$ scores must be at least $\binom{n-1}{2} = (n-1)(n-2)/2$, so $(n-1)y \geq (n-1)(n-2)/2$, so $n-2 \leq 2y$. We now have $n-2 \leq 2y < n-1$, with n and y being integers. This is only possible if $n = 2y+2$ (so n must be even and $y = n/2-1$). Substituting this back into the relation $x = (n-1)(n/2-y)$ gives $s_1 = x = n-1$, so player p_1 must beat all the other players. For $k < n$ we see that the sum of the last k of the terms s_i is ky and $y = (n-2)/2 \geq (k-1)/2$ so $ky \geq k(k-1)/2 = \binom{k}{2}$. Thus, Landau's Theorem tells us that a tournament with these scores does indeed exist. In fact, in Example 13.12 we described how to use modular arithmetic to produce a tournament with an odd number of players and all scores the same. We just need to add one champion player to get the scores considered here.

Exercise 5.6. Consider the following Latin rectangle:

$$\begin{bmatrix} 1 & 2 & 3 & \ldots & n \\ n & 1 & 2 & \ldots & n-1 \end{bmatrix}$$

In how many ways can it be extended to a $3 \times n$ Latin rectangle with entries from $\{1, 2, \ldots, n\}$? (You can convert this to a rook placement problem as in Problem 8.9. You should find that the relevant board is one that we have already discussed.)

Solution: Translating to rooks, we need the number of ways of placing n non-challenging rooks on an $n \times n$ board like this:



This is the complement of the board Q_n' in Definition 10.10. Proposition 10.11 gives

$$c_k(Q_n') = c_k(Q_n) + c_{k-1}(Q_{n-1}) = \binom{2n-k}{k} + \binom{2n-1-k}{k-1}.$$

Thus, Theorem 10.3 tells us that the number we need is

$$\sum_{k=0}^{n}(-1)^k(n-k)!\left(\binom{2-k}{k}+\binom{2n-1-k}{k-1}\right).$$

Exercise 5.7. Consider the following square, which has the variable x in the bottom right corner:

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 1 & 2 \\ 3 & 4 & 6 & 1 \\ 4 & 1 & 2 & x \end{bmatrix}$$

(a) For which values of x can L be extended to a 7×7 Latin square with entries $\{1,\dots,7\}$?
(b) For which values of x can L be extended to a 6×6 Latin square with entries $\{1,\dots,6\}$?
(c) For the value of x in (b), find one extension of the specified type.

Solution: Ignoring the x, we have the following multiplicities:

i	1	2	3	4	5	6	7
$m_L(i)$	4	3	2	3	1	2	0

By Theorem 14.22, a $p\times q$ Latin rectangle extends to an $n\times n$ Latin square with entries $\{1,\dots,n\}$ if and only if $m_L(i)\geq p+q-n$ for all i such that $1\leq i\leq n$.

(a) Taking $p=q=4$ and $n=7$ we need $m_L(i)\geq 4+4-7=1$ for each i such that $1\leq i\leq 7$. Thus, the extension to a 7×7 Latin square is possible if and only if we take $x=7$ so as to change $m_L(7)$ to 1.
(b) Taking $p=q=4$ and $n=6$, we need $m_L(i)\geq 4+4-6=2$ for each i such that $1\leq i\leq 6$. Thus the extension to a 6×6 Latin square is possible if and only we take $x=5$ so as to change $m_L(5)$ to 2.
(c) To find an extension, we first add a new row. The possible values for the four columns are $\{2,6\}$, $\{3,5\}$, $\{4,5\}$ and $\{3,6\}$. Note that $m_L(3)=m_L(5)=m_L(6)=2$, which is the minimum allowed value, so 3, 5 and 6 are barely plausible for the extension problem. Thus, we need to ensure that 3, 5 and 6 appear in the new row. We can do this by taking $(6,5,4,3)$ as the new row. There is then a unique possible way to add a sixth row, namely $(2,3,5,6)$. This gives the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 1 & 2 \\ 3 & 4 & 6 & 1 \\ 4 & 1 & 2 & 5 \\ 6 & 5 & 4 & 3 \\ 2 & 3 & 5 & 6 \end{bmatrix}$$

We now need to add another column. The possibilities for the six different rows are as follows:

$$a_1\in\{5,6\} \qquad a_2\in\{3,4\} \qquad a_3\in\{2,5\} \qquad a_4\in\{3,6\} \qquad a_5\in\{1,2\} \qquad a_6\in\{1,4\}.$$

If we choose $a_1=5$, we find that a_3 must be 2, so a_5 must be 1, so a_6 must be 4, so a_2 must be 3, so a_4 must be 6. Thus, column 5 must be $(5,3,2,6,1,4)$. There is now only one possibility for column 6, namely $(6,4,5,3,2,1)$. The final result is the following Latin square:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 3 & 4 & 6 & 1 & 2 & 5 \\ 4 & 1 & 2 & 5 & 6 & 3 \\ 6 & 5 & 4 & 3 & 1 & 2 \\ 2 & 3 & 5 & 6 & 4 & 1 \end{bmatrix}$$

Exercise 5.8. Let p,q and n be positive integers with $p\leq n$ and $q\leq n$. Let L be a $p\times q$ Latin rectangle in which each of the numbers $\{1,2,\dots,n\}$ occurs the same number of times. Show that L can be extended to an $n\times n$ Latin square.

Solution: We are given that $p, q \leq n$ and that all the numbers $m_L(i)$ are the same, say $m_L(i) = k$. The total number of entries in the rectangle is pq, but each of the entries $1, \ldots, n$ occurs k times, so we must have $pq = nk$. We can rewrite this as $k = pq/n$, so for all i we have

$$\begin{aligned} e_L(i) &= m_L(i) + n - p - q = pq/n + n - p - q \\ &= (n^2 - pn - qn + pq)/n = (n-p)(n-q)/n. \end{aligned}$$

As $p, q \leq n$ we see that $n - p, n - q \geq 0$ and so $e_L(i) \geq 0$ for all i. It follows from Theorem 79 that L can be extended to an $n \times n$ Latin square.

Exercise 5.9. Write down two orthogonal 3×3 Latin squares.

Solution: There are 36 different possible correct answers for this question. Here is the simplest one:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}$$

Exercise 5.10. Given integers v and k with $1 < k < v$ show that there exists a design with parameters $\left(v, \binom{v}{k}, \binom{v-1}{k-1}, k, \binom{v-2}{k-2}\right)$. What are the parameters in the special case where $v > 2$ and $k = v - 1$?

Solution: Put $V = \{1, \ldots, v\}$ and $b = \binom{v}{k}$ and $B = \{1, \ldots, b\}$. Note that b is the number of subsets of size k in V, so we can list those subsets as $C_1, \ldots, C_b$. Note that $|V| = v$ and $|B| = b$ and $|C_j| = k$ for all j. We now define $R_p = \{j \mid p \in C_j\} \subseteq B$ as usual. To choose a set C_j containing p, we start with $\{p\}$ and add in $k - 1$ further elements taken from the set $V \setminus \{p\}$, which has size $v - 1$. There are $\binom{v-1}{k-1}$ ways to do this, so $|R_p| = \binom{v-1}{k-1}$.

Similarly, suppose we have varieties $p \neq q$, and we want to choose an element of $R_p \cap R_q$, or in other words a set C_j containing both p and q. To do this, we start with $\{p, q\}$, and add in $k - 2$ further elements taken from the set $V \setminus \{p, q\}$, which has size $v - 2$. There are $\binom{v-2}{k-2}$ ways to do this, so $|R_p \cap R_q| = \binom{v-2}{k-2}$. The main claim is now clear.

Finally, when $k = v - 1$ we get

$$(v, b, r, k, \lambda) = \left(v, \binom{v}{v-1}, \binom{v-1}{v-2}, v - 1, \binom{v-2}{v-3}\right) = (v, v, v - 1, v - 1, v - 2).$$

Exercise 5.11. Suppose we have a design with parameters (v, b, r, k, λ). Prove that there is also a design with parameters $(v, b, b - r, v - k, b - 2r + \lambda)$.

[Hint: the idea is to replace each block by its complement in the set of varieties. You need to check that this does result in a design with the specified parameters.]

Solution: The original design consists of sets $C_j \subseteq V$ for $j \in B$, and the corresponding sets $R_p = \{j \mid p \in C_j\} \subseteq B$. These are assumed to have the following properties:

(a) $|V| = v$
(b) $|B| = b$
(c) $|R_p| = r$ for all $p \in V$
(d) $|C_j| = k$ for all $j \in B$
(e) $|R_p \cap R_q| = \lambda$ for all $p, q \in V$ with $p \neq q$.

The new design will have the same sets V and B (so axioms (a) and (b) remain valid) but the new column sets are

$$C'_j = V \setminus C_j = \{p \in V \mid p \notin C_j\}.$$

It is clear that $|C'_j| = |V| - |C_j| = v - k$ for all j. The corresponding row sets are

$$R'_p = \{j \mid p \in C'_j\} = \{j \mid p \notin C_j\} = \{j \mid j \notin R_p\} = B \setminus R_p.$$

It follows that

$$|R'_p| = |B| - |R_p| = b - r$$

for all p. For $p \neq q$ we also have

$$\begin{aligned}
|R_p \cap R_q| &= \lambda \\
|R_p \cup R_q| &= |R_p| + |R_q| - |R_p \cap R_q| = 2r - \lambda \\
R'_p \cap R'_q &= (B \setminus R_p) \cap (B \setminus R_q) = B \setminus (R_p \cup R_q) \\
|R'_p \cap R'_q| &= |B| - |R_p \cup R_q| = b - 2r + \lambda.
\end{aligned}$$

Thus, the sets C'_j give a block design with parameters $(v, b, b - r, v - k, b - 2r + \lambda)$

Exercise 5.12.

(a) Explain briefly how to construct a $(23, 23, 11, 11, 5)$ design.
(b) Show that, if a $(23, 23, r, k, \lambda)$ design exists, then $r = k$ and $k(k - 1) = 22\lambda$. Hence find all values of r, k and λ such that a $(23, 23, r, k, \lambda)$ design exists. (Remember that to be sure that one does exist you must explain briefly how to construct it. Exercises 5.10 and 5.11 will be useful for this.)

Solution:

(a) Note that 23 is prime, and of the form $4n+3$ where $n = 5$. Theorem 15.16 therefore gives us a block design with parameters $(4n+3, 4n+3, 2n+1, 2n+1, n) = (23, 23, 11, 11, 5)$. In more detail, we have $B = V = \mathbb{Z}/23$ and $C_j = j + Q$ where

$$Q = \{i^2 \mid i \in (\mathbb{Z}/23) \setminus \{0\}\}.$$

We can tabulate the values as follows:

i	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9	± 10	± 11
i^2	1	4	9	16	25	36	49	64	81	100	121
i^2 (mod 2)3	1	4	9	-7	2	-10	3	-5	-11	8	6

This gives

$$Q = \{1, 2, 3, 4, -5, 6, -7, 8, 9, -10, -11\}.$$

(b) Suppose we have a design with parameters $(v = 23, b = 23, r, k, \lambda)$. Putting $v = b = 23$ in Proposition 15.4 we get

$$23k = 23r \qquad 23k(k-1) = \lambda \times 23 \times 22 \qquad r(k-1) = 22\lambda.$$

This simplifies easily to $r = k$ with $k(k - 1) = 22\lambda$. From the definition of a block design we also have $0 < k < 23$ and $\lambda > 0$. As $\lambda > 0$, the relation $k(k - 1) = 22\lambda$ implies $k > 1$. Also, the relation $k(k - 1) = 22\lambda$ implies that $k(k - 1)$ is divisible by 11, and 11 is prime, so either k or $k - 1$ must be divisible by 11. From this we see that $k \in \{11, 12, 22\}$. Once we know k, the full list of parameters is

$$(v, b, r, k, \lambda) = (23, 23, k, k, k(k - 1)/22).$$

(1) If $k = 11$ the parameters are $(23, 23, 11, 11, 5)$. We produced a design with these parameters in part (a).
(2) If $k = 12$ the parameters are $(23, 23, 12, 12, 6)$. Note that if $(v, b, r, k, \lambda) = (23, 23, 11, 11, 5)$ then

$$(v, b, b - r, v - k, b - 2r + \lambda) = (23, 23, 12, 12, 6).$$

Thus, we can take the design from (a) and apply Exercise 5.11 to it to get the required design with $k = 12$.
(3) If $k = 22$ the parameters are $(23, 23, 22, 22, 21)$. This corresponds to the case of Exercise 5.10 where $v = 23$ and $k = 22$.

Exercise 5.13.

(a) Show that there cannot be a design with $k = 3$, $\lambda = 1$ and $v = 11$.
(b) Show that if a design has $k = 3$ and $\lambda = 1$, then v must be congruent to 1 or 3 mod 6.

Solution:

(a) In any design, we have $\frac{bk}{v} = \frac{\lambda(v-1)}{k-1}$. So for such a design b must satisfy $\frac{3b}{11} = \frac{1.(11-1)}{(3-1)}$. That is, $3b = 55$. But this is impossible since b must be an integer.

(b) We have $r(k-1) = \lambda(v-1)$, with $k = 3$ and $\lambda = 1$. So $2r = v - 1$ and so v is odd. So v is $1, 3$ or 5 mod 6. On the other hand, $b(k-1)k = \lambda(v-1)v$ gives $6b = (v-1)v$. So 3 divides v or 3 divides $v - 1$. Hence v must be 1 or 3 mod 6.

Exercise 5.14. Consider an odd number $n = 2m+1$ with $m \geq 2$. (You could take $n = 7$ to make the problem more concrete.) We could try to define a block design by taking $B = V = \mathbb{Z}/n$ and $C_j = \{j+1, \dots, j+m\}$ for all j (where the additions are all done modulo n). Explain why this does not actually give a block design.

Solution: The corresponding row sets are

$$R_p = \{j \mid p \in C_j\} = \{j \mid p = j + i \text{ for some } i \in \{1, \dots, m\}\} = \{p-1, \dots, p-m\}.$$

We thus have $|V| = |B| = n$, and $|C_j| = m$ for all j, and $|R_p| = m$ for all p. However, we have

$$R_0 \cap R_1 = \{-1, \dots, -m\} \cap \{0, \dots, -(m-1)\} = \{-1, \dots, -(m-1)\}$$
$$R_0 \cap R_2 = \{-1, \dots, -m\} \cap \{1, \dots, -(m-2)\} = \{-1, \dots, -(m-2)\},$$

so $|R_0 \cap R_1| = m - 1$ and $|R_0 \cap R_2| = m - 2$. As the intersections $R_p \cap R_q$ do not all have the same size, we do not have a block design.

More concretely, when $n = 7$ we have $m = 3$ and

$$R_0 \cap R_1 = \{-1, -2, -3\} \cap \{0, -1, -2\} = \{-1, -2\}$$
$$R_0 \cap R_2 = \{-1, -2, -3\} \cap \{1, 0, -1\} = \{-1\}.$$