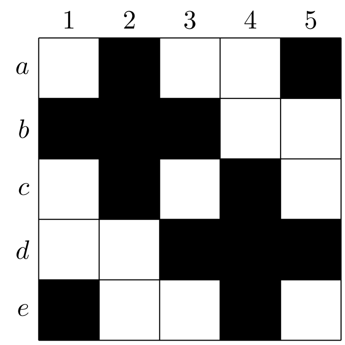
MAS334 COMBINATORICS — PROBLEM SHEET 4 — Solutions

Please hand in exercises 4.1 and 4.4 by the end of Week 9.

Exercise 4.1. Use the tabular method to find all full matchings for the following board:

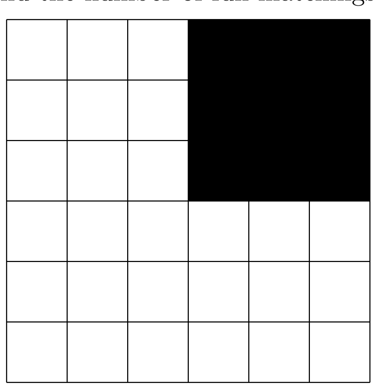


Solution: The table is as follows:

$a1$	$b4$	$c3$	$\underline{d2}$	$\underline{e5}$	✓
		$\underline{c5}$	$\underline{d2}$	$\underline{e3}$	✓
	$\underline{b5}$	$\underline{c3}$	$\underline{d2}$		✗
$a3$	$b4$	$c1$	$\underline{d2}$	$\underline{e5}$	✓
		$\underline{c5}$	$d1$	$\underline{e2}$	✓
			$\underline{d2}$		✗
	$\underline{b5}$	$\underline{c1}$	$\underline{d2}$		✗
$\underline{a4}$	$\underline{b5}$	$c1$	$\underline{d2}$	$\underline{e3}$	✓
		$\underline{c3}$	$d1$	$\underline{e2}$	✓
			$\underline{d2}$		✗

This gives 6 full matchings.

Exercise 4.2. Use Remark 10.7 to find the number of full matchings for the following board B:



Can you explain the answer in a different way?

Solution: The complementary board $\overline{B}$ is just a copy of F_3, giving $c_k(\overline{B}) = \binom{3}{k}^2 k!$, or more explicitly

$$c_0(\overline{B}) = 1 \qquad c_1(\overline{B}) = 9 \qquad c_2(\overline{B}) = 18 \qquad c_3(\overline{B}) = 6.$$

We also have $c_k(\overline{B}) = 0$ for $k > 3$. From this we get

$$\begin{aligned} c_6(B) &= 6!c_0(\overline{B}) - 5!c_1(\overline{B}) + 4!c_2(\overline{B}) - 3!c_3(\overline{B}) \\ &= 720 \times 1 - 120 \times 9 + 24 \times 18 - 6 \times 6 = 36. \end{aligned}$$

This can also be explained as follows. We need to place 3 rooks in the first 3 rows. Clearly, these can only appear in the first three columns, and in fact they must fill the first three columns, so there are no free places remaining in the bottom left quarter of the board. We then need to place three rooks in the last three rows, and these are forced to go in the bottom right quarter. Thus, to choose a full matching for the whole board, we just need a full matching for the top left copy of F_3, and a full matching for the bottom right copy of F_3. The number of possibilities is therefore $3! \times 3! = 36$.

Exercise 4.3. Let B be an $n \times n$ board (with $n > 0$), and let $\overline{B}$ be the complement. Theorem 10.3 and Remark 10.7 tell us that

$$\begin{aligned} c_n(B) &= \sum_{k=0} (-1)^k (n-k)! c_k(\overline{B}) \\ c_n(\overline{B}) &= \sum_{k=0} (-1)^k (n-k)! c_k(B). \end{aligned}$$

Check these equations directly in the case where B is the full board F_n.

Solution: By Proposition 7.11, we have

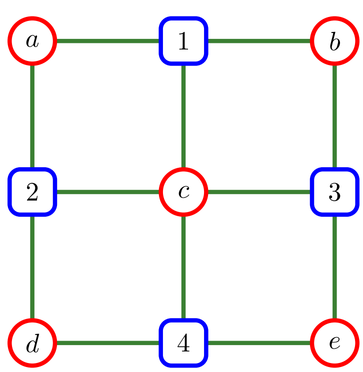
$$c_k(B) = \binom{n}{k}^2 k! = \frac{n!^2}{k!^2\,(n-k)!^2} k! = \frac{n!^2}{k!\,(n-k)!^2}.$$

In particular, we have $c_n(B) = n!$. On the other hand, $\overline{B}$ has all squares black, so no rooks can be placed there. This means that $c_k(\overline{B}) = 0$ for all $k > 0$, and in particular $c_n(\overline{B}) = 0$, but we still have $c_0(\overline{B}) = 1$. Thus, in the sum $\sum_{k=0} (-1)^k (n-k)! c_k(\overline{B})$ we see that all terms are zero except for the $k = 0$ term which is $n!$, so the sum is the same as $c_n(B)$ as expected. For the other sum we have

$$\begin{aligned} \sum_{k=0} (-1)^k (n-k)! c_k(B) &= \sum_{k=0} (-1)^k (n-k)! \frac{n!^2}{k!\,(n-k)!^2} = \sum_{k=0} (-1)^k \frac{n!^2}{k!\,(n-k)!} \\ &= n! \sum_{k=0} n (-1)^k \binom{n}{k} = n!(1 + (-1))^n = n!\,0^n = 0. \end{aligned}$$

Again, this is the same as $c_n(\overline{B})$, as expected.

Exercise 4.4. Consider the matching problem where $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$ and the incidence graph is as follows:



(a) Find a very plausible subset $U \subseteq B$.
(b) Find a barely plausible subset $V \subseteq B$ with $V \neq \emptyset$.

(c) Does the matching problem have a solution?

Solution:

(a) We have $C_{\{a\}} = \{1, 2\}$, so $|C_{\{a\}}| > |\{a\}|$, so the set $U = \{a\}$ is very plausible. (In fact, one can check that if U is any subset of B with $1 \leq |U| \leq 3$ then U is very plausible.)

(b) We have $C_{\{a,b,c,d\}} = \{1, 2, 3, 4\}$, so the set $V = \{a, b, c, d\}$ is barely plausible. (In fact, one can check that if V is any subset of B with $|V| = 4$ then V is barely plausible.)

(c) As we have 5 jobs and only 4 people, it is clear that the problem is not solvable. As another way to say this, the full set B has $C_B = A$, so $|B| = 5$ and $|C_B| = 4$, so the set B is implausible, so Lemma 11.5 guarantees that there is no solution.

Exercise 4.5. Consider a matching problem where $B = \{1, 2, \ldots, n\}$ and $|C_k| \geq k$ for all k. Prove that the problem is solvable.

Solution: Consider a subset $U \subseteq B$, with $|U| = p$ say. If $p = 0$ then $U = \emptyset$ and $C_U = \emptyset$ so U is plausible. Suppose instead that $p > 0$, and let m be the largest element of U. As all p elements of U are less than or equal to m, we must have $m \geq p$. We also have $C_U \supseteq C_m$ and so $|C_U| \geq |C_m| \geq m \geq p = |U|$, so we again see that U is plausible. As U was arbitrary, we can now apply Hall's Theorem to see that a solution exists.

Exercise 4.6. Consider a job allocation problem where $|A| = |B| = n$, and every subset $U \subseteq B$ is barely plausible. What can we conclude?

Solution: We can list the elements of B as $\{b_1, \ldots, b_n\}$ (with $b_1, \ldots, b_n$ all being different). The set $\{b_i\}$ is barely plausible, so $|C_{b_i}| = 1$. Thus, there is only one person who is qualified to do job b_i. Call that person a_i. We now have people $a_1, \ldots, a_n$, but we do not yet know that these people are all different; perhaps there is one multitalented person who can do every job, and everyone else is completely incompetent. However, we also know that the full set B is barely plausible, so $|C_B| = |B| = n$. Here

$$C_B = C_{b_1} \cup \cdots \cup C_{b_n} = \{a_1\} \cup \cdots \cup \{a_n\} = \{a_1, \ldots, a_n\}.$$

As this set has size n, the elements $a_1, \ldots, a_n$ must all be different. As $|A| = n$, it follows that these are all the elements of A. Thus, we have a one-to-one correspondence between jobs and people, with each person corresponding to the only job that they can do, and each job corresponding to the only qualified candidate.