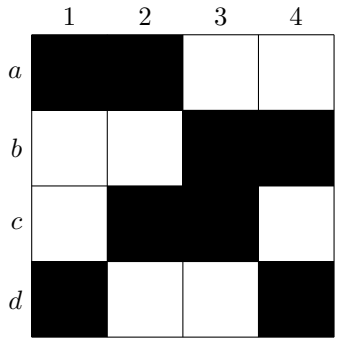
MAS334 COMBINATORICS — PROBLEM SHEET 3

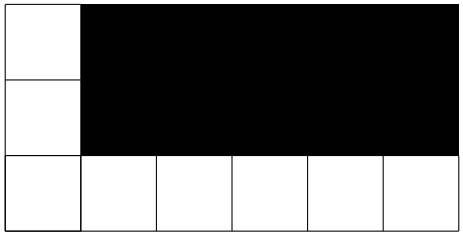
Please hand in exercises 3.1 and 3.2 by the end of Week 6.

Exercise 3.1. Consider the following board:



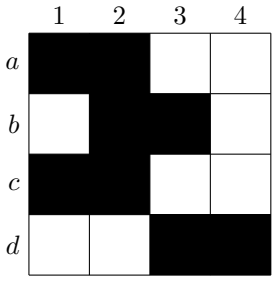
(a) Draw the corresponding incidence graph.
(b) Write down the row sets R_x and column sets C_y.
(c) There are two possible full matchings. Draw the corresponding rook placements and incidence graphs as in Example 6.7.

Exercise 3.2. Consider an L-shaped board B, with p squares on the left edge and q squares on the bottom edge. The case $(p, q) = (3, 6)$ is illustrated below.

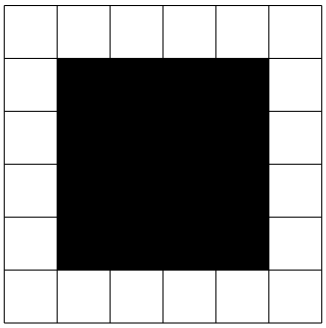


What is the rook polynomial? (Give a formula for all p and q.)

Exercise 3.3. Find the rook polynomial of the following board B:

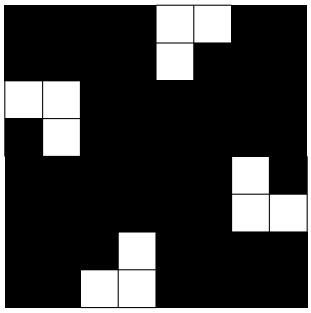


Exercise 3.4. Consider a ring-shaped board R_n of size $n \times n$ with $n \geq 3$, as illustrated below for the case $n = 6$. What is the rook polynomial?



It is best to start by thinking about rooks in the corner positions, and then think about what else we can do after the corner rooks are in place. The answer comes out as a moderately complicated expression, which can be expanded and simplified. If you wish, you can ignore the expansion and simplification step, or get a computer to do it.

Exercise 3.5. What is the rook polynomial of the following board?



Exercise 3.6. Let n be a positive even integer and consider an $n \times n$ chess-board in which the squares are coloured black and white in the usual chequered fashion. In how many ways can n non-challenging rooks be placed on the white squares?

Exercise 3.7. Which of the following polynomials can be the rook polynomial of a board? Give reasons, including examples of appropriate boards, where possible.

$$p_1(x) = 1 + x$$
$$p_2(x) = (1 + x)^n$$
$$p_3(x) = 1 + 4x + 2x^2$$
$$p_4(x) = (1 + 4x + 2x^2)^2$$
$$p_5(x) = 1 - 3x$$
$$p_6(x) = 1 + 2x + 2x^2.$$