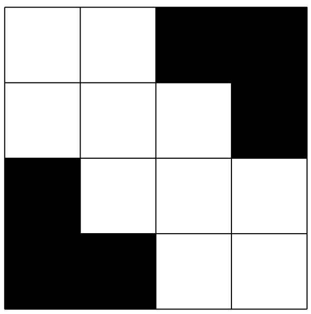
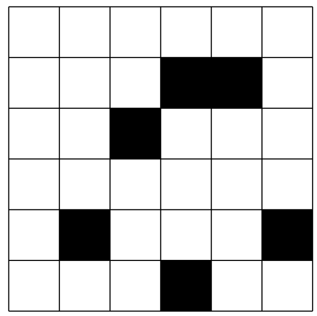
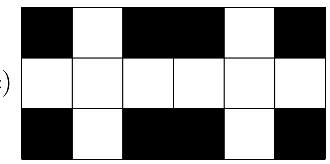
MAS334 COMBINATORICS — PROBLEM SHEET 2

Please hand in exercises 2.1 and 2.5 by the end of Week 4.

Exercise 2.1. Consider the following boards:

(a) 

(b) 

(c) 

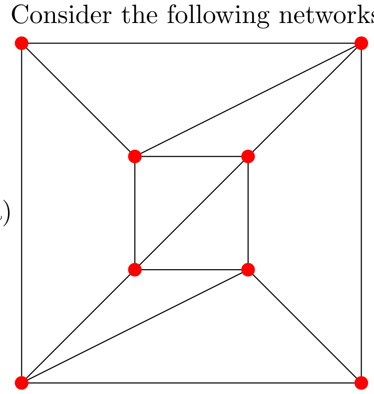
(The black squares do not count as part of the boards.) Which of these can be covered by disjoint dominos?

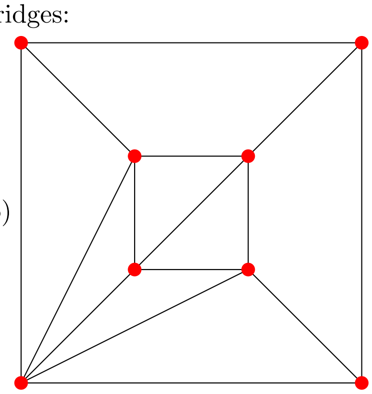
Exercise 2.2. On an $n \times n$ board there are n^2 chess pieces, one on each square. I wish to move each piece to an adjacent square (in the same row or column) so that after all the n^2 pieces have moved there is still one piece on each square.

(a) Show that this can be done if n is even. (Give explicit instructions for where to move each piece, not just an abstract argument to suggest that it is possible.)

(b) By a colourful argument show that it cannot be done if n is odd. (It is not enough to try some plausible approach and show that that approach fails. There might be some complicated and nonobvious pattern of movement that does the job. You need to give a proper proof to show that that cannot happen.)

Exercise 2.3. Consider the following networks of nodes and bridges:

(a) 

(b) 

In each case, say whether it is possible to have a circular tour that crosses each bridge precisely once, and also whether it is possible to have a non-circular tour that crosses each bridge precisely once.

Exercise 2.4. Given any positive integer $n \geq 1$, if we divide by two repeatedly we will eventually get to an odd number. Thus, we have $n = 2^t m$ for some $t \geq 0$ and some odd number m. We call m the *odd part* of n. For example, $60 = 2^2 \times 15$, so the odd part of 60 is 15.

Show that, given any $n+1$ different positive integers less than or equal to $2n$, there will exist two with the same odd part. Deduce that one of those numbers is a multiple of the other.

Exercise 2.5. Let n be a positive integer. By considering numbers of the form $1, 11, 111, \dots$ and their remainders modulo n, show that there exists a number of the form $11\dots10\dots00$ which is a multiple of n.

Exercise 2.6. Prove that there exist two different powers of 7 whose difference is divisible by 1000.

Exercise 2.7. Let $a_1, \dots, a_{50}$ be points in the unit square $[0,1]^2$. Show that there are indices $i < j$ such that the distance from a_i to a_j is less than 0.21. (Hint: divide the square into small boxes and apply the pigeonhole principle.)

Exercise 2.8. Using the pigeonhole principle, explain why there is no compression algorithm that can compress every possible $8MB$ file down to $1MB$ in such a way that it can be uncompressed without errors.

Exercise 2.9. Lemma 5.4 says that for a finite, nonempty set I we have $\sum_{J\subseteq I}(-1)^{|J|} = 0$. Check this explicitly in the case $I = \{a, b, c\}$.

Exercise 2.10. Suppose that we choose a number $k \in \{0, 1, \dots, 999\}$ at random. What is the probability that k and 1000 are coprime?

Exercise 2.11. Suppose we have a set B with subsets $B_1, \dots, B_4$ such that $B = B_1 \cup \dots \cup B_4$ and $|B|$ is odd. Suppose there are numbers n_1, n_2, n_3 such that

- $|B_i| = n_1$ for all i
- $|B_{ij}| = n_2$ for all $i < j$
- $|B_{ijk}| = n_3$ for all $i < j < k$.

Prove that $|B_{1234}|$ is odd.

Exercise 2.12. Consider the set $X = [2, 1000] = \{2, 3, 4, \dots, 1000\}$. For each prime p, let X_p be the subset of numbers in X that are divisible by p. As in the Inclusion-Exclusion Principle, we write $X_{p,q,r}$ for $X_p \cap X_q \cap X_r$, and so on. We will investigate the sizes of some sets related to these. It will be helpful to use the notation

$$\lfloor x \rfloor = \text{ largest integer } n \text{ such that } n \leq x$$

(so for example $\lfloor 7.01 \rfloor = \lfloor 7.89 \rfloor = \lfloor 7.77 \rfloor = 7$).

(a) Show that $|X_2| = 5000$ and $|X_{2,3}| = 1666$.
(b) Give a formula for $|X_{p,q,r}|$.
(c) Using the IEP, show that there are precisely 7334 numbers in X that are divisible by at least one of the primes 2, 3 and 5. Of these 7334 numbers, show that 3 are prime and 7331 are not prime.
(d) Give an upper bound for the number of primes in X. How would this bound change if we tested for divisibility by 7 as well as 2, 3 and 5?

Exercise 2.13. Let A and B be finite sets, with $|A| = m$ and $|B| = n$ say. We will assume that $m \geq n$. Let F be the set of all functions from A to B. For each $b \in B$, let $F_b \subseteq F$ be the subset of functions $f\colon A \to B$ such that $b \notin f(A)$. We also let $E \subseteq F$ be the set of surjective functions.

(a) Explain why $|F| = n^m$.

(b) Show that $|F_b| = (n-1)^m$ for all $b \in B$.
(c) Show that for $b \neq c$ in B we have $|F_b \cap F_c| = (n-2)^m$.
(d) Give a formula for E in terms of the sets F_b.
(e) By applying the negative IEP to your formula in (d), prove that

$$|E| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^m.$$

Exercise 2.14.

(a) Explain briefly why there are $\binom{n-1}{k-1}$ positive integer solutions of the equation

$$x_1 + x_2 + \cdots + x_k = n.$$

(b) Use the Inclusion/Exclusion Principle to find the number of positive integer solutions of the equation

$$x_1 + x_2 + x_3 = 20$$

satisfying the conditions $x_1 \leq 5$, $x_2 \leq 10$ and $x_3 \leq 15$.

Exercise 2.15. Let P be the set of permutations of $\{1, \ldots, 9\}$. Let Q be the subset of permutations $\sigma \in P$ satisfying $\sigma(i) + i \leq 10$ for all i. Let R be the subset of permutations $\sigma \in P$ satisfying $\sigma(i) = i \pmod 3$ for all i. Find $|P|$, $|Q|$ and $|R|$.

Exercise 2.16. Put $B = \{1, \ldots, n\}$ and $B_i = \{1, \ldots, i\} \subseteq B$ for $i = 1, \ldots, n$. Put $B' = B_1 \cup \cdots \cup B_n$. What is $|B'|$? (The answer is obvious.) Check that the IEP gives the same as the obvious answer. (Hint: group the nonempty subsets $I \subseteq \{1, \ldots, n\}$ according to their minimum elements.)

Exercise 2.17. Put $B = \{0, 1, \ldots, 15\}$. Any number $k \in B$ can be expressed in base 2 with four binary digits, for example

$$11 = 8 + 2 + 1 = 1.2^3 + 0.2^2 + 1.2^1 + 1.2^0 = 1011_2.$$

Let B_i be the subset of those $k \in B$ such that the binary expansion contains 2^i. (For example, the expansion of 11 contains 2^3 but not 2^2, so $11 \in B_3$ but $11 \notin B_2$.) As usual, we put $B^* = B \setminus (B_0 \cup B_1 \cup B_2 \cup B_3)$. What is $|B^*|$? (The answer is easy.) Check that the IEP gives the same as the easy answer.