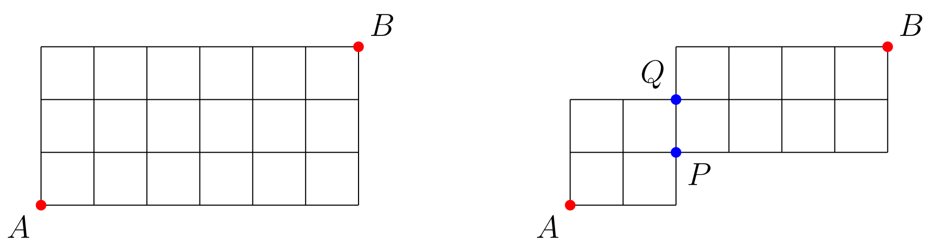
SCHOOL OF MATHEMATICS AND STATISTICS — **Autumn Semester 2022–23**

Combinatorics — **3 hours**

Attempt all the questions. Give justification for all numerical answers. The allocation of marks is shown in brackets.

1 Consider the following diagram:



We are interested in paths through this grid from A to B (with each path consisting of steps of length one upwards or to the right, as usual).

(a) How many paths are there from A to B in the left hand diagram? ***(3 marks)***

(b) How many paths are there from A to B in the right hand diagram?

[Hint: Consider whether such paths pass through P or Q or both or neither.] ***(7 marks)***

2 Find the number of integer solutions for each of the following problems:

(a) $x_1 + \cdots + x_9 = 15$ with $x_1, \ldots, x_9 \geq 0$. ***(3 marks)***

(b) $x_1 \times \cdots \times x_9 = 15$ with $x_1, \ldots, x_9 \geq 0$. ***(3 marks)***

(c) $x_1 + \cdots + x_9 = 5 \pmod{10}$ with $0 \leq x_1, \ldots, x_9 < 10$. ***(3 marks)***

(d) $x_1^2 + \cdots + x_9^2 = 3$ with $x_1, \ldots, x_9 \in \mathbb{Z}$. ***(3 marks)***

3 Let $k \geq 1$ and $n \geq 3k-1$. This question concerns seating k couples in a row of n seats. The couples are $c_i = (c_i^L, c_i^R)$ for $1 \leq i \leq k$ and we want to seat them according to the following rules.

- For each i, c_i^L sits in the adjacent seat to the left of c_i^R.
- The couples are in order $c_1, c_2, \ldots, c_k$ from left to right.
- Different couples are not adjacent: there is a gap of at least one seat between one couple and another.

Let T_n^k denote the number of ways this can be done.

(a) What is T_n^k? Give a direct argument for your answer. ***(4 marks)***

(b) Show, from the description of the seating problem, that $T_{3k-1}^k = 1$ and that $T_n^k = T_{n-1}^k + T_{n-3}^{k-1}$ for $n \geq 3k$. ***(5 marks)***

(c) Check that your answer to part (i) is consistent with (ii). ***(3 marks)***

4 (a) State the Pigeonhole Principle. ***(2 marks)***

(b) Let X be a set of 11 numbers from $\{1, 2, \ldots, 80\}$. Show that there exist two different subsets of X each having exactly 4 elements and such that the sum of their elements is the same. ***(5 marks)***

5 (a) State the positive form of the Inclusion/Exclusion Principle. ***(3 marks)***

(b) Use the Inclusion/Exclusion Principle to find the number of permutations of the numbers $1, 2, \ldots, 10$ such that at least one even number is fixed.
(7 marks)

6 (a) Let $n \geq 3$. Find the rook polynomial of the full $n \times 3$ board. ***(4 marks)***

(b) Which of the following polynomials can be the rook polynomial of a board? Give reasons for your answers, including examples of appropriate boards.

(i) $1 - 7x$.

(ii) $(1+x)(1+4x+2x^2)^2$.

(iii) $1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n$. ***(5 marks)***

7 Recall that a derangement of $\{1, 2, \ldots, n\}$ is a permutation leaving none of the numbers fixed. We write d_n for the number of derangements of $\{1, 2, \ldots, n\}$.

(a) Show that

$$\sum_{k=0}^{n} \binom{n}{k} d_{n-k} = n!.$$

(4 marks)

(b) Show that, for $n \geq 3$,

$$d_n = (n-1)(d_{n-2} + d_{n-1}).$$

(6 marks)

8 Suppose that we have two tournaments, each of $2n$ players, where the scores are T_i and U_i, for $1 \leq i \leq 2n$. Show that there is a tournament of $4n$ players with scores $T_i + n$, $U_i + n$, for $1 \leq i \leq 2n$. ***(5 marks)***

9 (a) Explain what it means for two $n \times n$ Latin squares with $P = Q = N = \{1, 2, \ldots, n\}$ to be orthogonal. ***(2 marks)***

(b) Prove that there exist at most $n - 1$ mutually orthogonal $n \times n$ Latin squares. ***(8 marks)***

10 (a) In a (v, b, r, k, λ)-block design, the number of varieties is v and the number of blocks is b. Explain the meaning of each of the other parameters. ***(3 marks)***

(b) State two equations relating r to the other parameters of a design. ***(2 marks)***

(c) Consider all choices of 3 numbers from $\{1, 2, \ldots, 6\}$. Show that these form the blocks of a design and determine the parameters. ***(5 marks)***

(d) Let $2 \leq i \leq n$. Show that there is a design with parameters

$$\left(n, \binom{n}{i}, \binom{n-1}{i-1}, i, \binom{n-2}{i-2}\right).$$

(5 marks)

End of Question Paper