Combinatorics Exam Solutions 2019-20

(1) Put $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and consider subsets $U \subseteq N$.

(a) How many subsets are there in total? **(1 marks)**

(b) How many subsets U are there such that U contains at least one odd number? **(2 marks)**

(c) How many subsets U are there such that $|U|$ is odd? **(2 marks)**

(d) How many subsets U are there such that $U \neq \emptyset$ and $\max(U)$ is even? **(3 marks)**

Solution: Part (a) is standard, the rest is similar to problems that have been seen.

(a) The total number of subsets is $2^9 = 512$. [1]

(b) There are 4 even numbers in N (namely $2, 4, 6, 8$), so there are $2^4 = 16$ subsets that contain only even numbers. By subtracting this from the answer in (a), we see that there are $512 - 16 = 496$ subsets that contain at least one odd number. [2]

(c) To choose a set U such that $|U|$ is odd, we can choose an arbitrary subset $U_0 \subseteq \{1, \ldots, 8\}$, and then take $U = U_0$ if $|U_0|$ is odd, or $U = U_0 \cup \{9\}$ if $|U_0|$ is even. From this it is clear that the number of possibilities is $2^8 = 256$. Alternatively, the number is

$$\binom{9}{1} + \binom{9}{3} + \binom{9}{5} + \binom{9}{7} + \binom{9}{9} = 9 + 84 + 126 + 36 + 1 = 256.[2]$$

(d) To form a subset U with $\max(U) = 8$, we take $\{8\}$ and throw in an arbitrary subset of $\{1, \ldots, 7\}$, for which there are 2^7 possibilities. Similarly, there are 2^5 subsets with $\max(U) = 6$, and 2^3 with $\max(U) = 4$, and 2 with $\max(U) = 2$. Thus, the total number is $2 + 2^3 + 2^5 + 2^7 = 2 + 8 + 32 + 128 = 170$. [3]

(2) State and prove Pascal's relation for binomial coefficients. **(5 marks)**

Solution: Bookwork Pascal's relation says that for $0 < k < n$ we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ [1]. Indeed, $\binom{n}{k}$ is the number of subsets $A \subseteq \{1, \ldots, n\}$ such that $|A| = k$ [1]. These subsets can be divided into two groups: those that do not contain n, and those that do contain n [1]. The first group just consists of the subsets of size k in $\{1, \ldots, n-1\}$, so there are $\binom{n-1}{k}$ of them [1]. The subsets in the second group are generated as follows: we choose $A_0 \subseteq \{1, \ldots, n-1\}$ with $|A_0| = k - 1$, then take $A = A_0 \cup \{n\}$. There are $\binom{n-1}{k-1}$ choices for A_0, and thus $\binom{n-1}{k-1}$ possibilities for A [1]. This gives $\binom{n-1}{k} + \binom{n-1}{k-1}$ possibilities altogether, so $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

An algebraic proof using factorials is also acceptable.

In the exam, many students gave an answer based on comparing the binary expansions of $(1 + x)^n$ and $(1+x)(1+x)^{n-1}$. I gave full marks for this but it is not really very satisfactory: Pascal's relation is easier and more fundamental than the binomial expansion, so it does not really make sense to deduce the former from the latter.

(3) Consider the equation $x_1 + x_2 + \cdots + x_{10} = 16$.

(a) How many solutions are there with $1 \leq x_i$ for all i? **(2 marks)**

(b) How many solutions are there with $1 \leq x_i \leq 2$ for all i? **(2 marks)**

(c) How many solutions are there with x_i odd and positive for all i? **(3 marks)**

Solution: Part (a) is standard, the rest is similar to problems that have been seen.
The number of variables is $k = 10$, and the right hand side is $m = 16$.

(a) The number of solutions is $\binom{m-1}{k-1} = \binom{15}{9} = 5005$. [2]

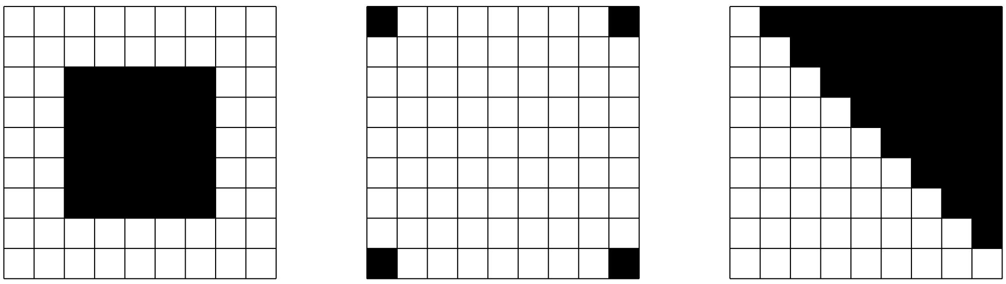
(b) Here six of the variables x_i must be equal to 2, and the rest must be equal to one [1]. The total number of possibilities is $\binom{10}{6} = 210$ [1].

(c) Here we must have $x_i = 2y_i + 1$ with $y_i \geq 0$ and

$$(2y_1 + 1) + \cdots + (2y_{10} + 1) = 16.\text{[1]}$$

There are 10 extra ones on the left hand side, so this is equivalent to $2\sum_i y_i = 6$ or $\sum_i y_i = 3$ [1]. As the variables y_i are allowed to be zero, the appropriate formula for the number of solutions is $\binom{10+3-1}{10-1} = \binom{12}{9} = 220$ [1].

(4) Consider $n \times n$ boards as illustrated below for the case $n = 9$: in A_n the middle $(n-4) \times (n-4)$ square is black, in B_n the four corners are black, and in C_n everything above and to the right of the diagonal is black. We will investigate whether they can be covered by disjoint dominos. Some answers will depend on n, and some will not. We will always assume that $n \geq 5$.



A_9 B_9 C_9

(a) Can A_n be covered by disjoint dominos? **(2 marks)**

(b) Can B_n be covered by disjoint dominos? **(4 marks)**

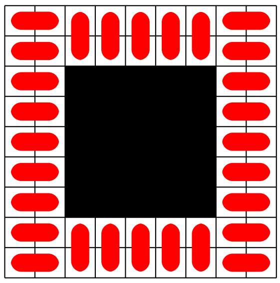
(c) Suppose we colour C_n with alternating white and grey squares in the usual chessboard pattern, with the bottom left square being grey. By considering diagonal stripes, find the number of white squares and grey squares (this will depend on whether n is odd or even). **(6 marks)**

(d) Can C_n be covered by disjoint dominos? **(1 marks)**

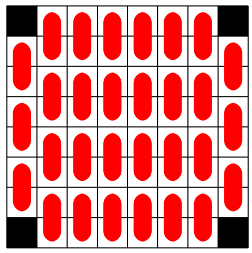
Solution: Parts (a) and (b) are completely standard, part (c) is a little harder, the deduction of (d) from (c) is standard.

(a) We can always cover A_n, with n horizontal dominos covering the first two columns, another n horizontal dominos covering the last two columns, and vertical dominos everywhere else, as shown below. This does not depend on the parity of n. [2] (Note that it is not enough to show that A_n has an even number of squares, or that it has the same number of white and grey squares; you need to specify how the dominos are laid out.)

(b) The number of squares in B_n is $n^2 - 4$. If n is odd then there is an odd number of squares, so B_n cannot be covered by disjoint dominos [2]. If n is even then B_n can be covered with disjoint vertical dominos as illustrated below. [2]



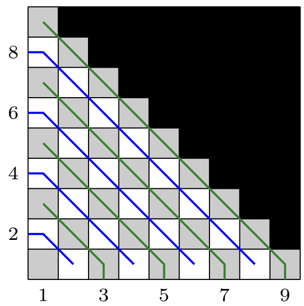
A_9



B_8

(Again, it is not enough to show that B_{2m} has an even number of squares, or that it has the same number of white and grey squares; you need to specify how the dominos are laid out.)

(c) Let w_n be the number of white squares in C_n, and let g_n be the number of grey squares. The white squares in C_n can be divided into stripes of even length as shown, and the grey squares can be divided into stripes of odd length. [1]



C_9

We find that g_n is the sum of all odd numbers less than or equal to n, and that w_n is the sum of the even numbers in the same range [1]. If $n = 2m$ we can use the standard arithmetic progression formula to get

$$g_n = 1 + 3 + \cdots + (2m-1) = m \times \frac{1 + 2m - 1}{2} = m^2 = \frac{n^2}{4}$$

$$w_n = 2 + 4 + \cdots + 2m = m \times \frac{2 + 2m}{2} = m(m+1) = \frac{n(n+2)}{4} [2]$$

If $n = 2m + 1$ we instead get

$$g_n = 1 + 3 + \cdots + (2m+1) = (m+1) \times \frac{1 + 2m + 1}{2} = (m+1)^2 = \frac{(n+1)^2}{4}$$

$$w_n = 2 + 4 + \cdots + 2m = m \times \frac{2 + 2m}{2} = m(m+1) = \frac{n^2 - 1}{4}. [2]$$

(d) In all cases we see that $w_n \neq g_n$, so C_n cannot be covered by disjoint dominos. [1]

(5)

(a) State the inclusion-exclusion principle. **(3 marks)**

(b) Let B be a finite set with $|B| = n$. Suppose we have subsets $B_i \subset B$ for $i = 1, \dots, m$, such that $|B_{i_1} \cap \dots \cap B_{i_r}| = n/3^r$ for all $i_1, \dots, i_r$ with $1 \leq r \leq m$ and $i_1 < \dots < i_r$. Give a fully simplified formula for $|B_1 \cup \dots \cup B_m|$. **(5 marks)**

Solution: Part (a) is bookwork. Part (b) has some overlap with questions that have been seen, but will require some insight.

(a) Consider a finite set B, with a family of subsets $B_a \subseteq B$ indexed by another finite set A. For $I \subseteq A$ put $B_I = \bigcap_{a \in I} B_a$, with the convention that $B_\emptyset = B$. Let B' be the union of the sets B_a, and put $B^* = B \setminus B'$. The IEP says that

$$|B'| = \sum_{I \neq \emptyset} (-1)^{|I|-1} |B_I|, [3]$$

or equivalently

$$|B^*| = \sum_I (-1)^{|I|} |B_I|.$$

Full marks will be given for either of the equivalent forms. Versions with ellipses instead of summation notation will be accepted if they are sufficiently clear.

(b) The negative form of the IEP involves terms $(-1)^{|I|}|B_I|$. We are given that $|B_I| = n/3^{|I|}$, so this can be written as $n.(-1/3)^k$, where $k = |I|$ [1]. If we fix k then there are $\binom{m}{k}$ possible choices for the set I [1]. Using the IEP and the binomial theorem we therefore get

$$|B^*| = \sum_I (-1)^{|I|} |B_I| = n \sum_{k=0}^{m} \binom{m}{k} (-1/3)^k [1] = n\left(1 - \tfrac{1}{3}\right)^m = n.(2/3)^m. [1]$$

For example, when $n = 3$ this is

$$|B^*| = \begin{array}{l} |B| \\ -|B_1| - |B_2| - |B_3| \\ +|B_{12}| + |B_{13}| + |B_{23}| \\ -|B_{123}| \end{array} = \begin{array}{l} n \\ -3n/3 \\ +3n/3^2 \\ -n/3^3 \end{array} = n\left(1 - 3.(\tfrac{1}{3}) + 3.(\tfrac{1}{3})^2 + (\tfrac{1}{3})^3\right) = n.(1 - \tfrac{1}{3})^3 = n.(\tfrac{2}{3})^3.$$

However, we were asked for $|B'|$, which is $|B| - |B^*|$, or

$$|B'| = n.(1 - (2/3)^m). [1]$$

Note that the calculation of $|B^*|$ is quite similar to the calculation of the number of derangements in the lecture notes.

(6)

(a) State the Pigeonhole Principle. **(2 marks)**

(b) Suppose we have a square with sides of length r. What is the maximum possible distance between two points in the square? **(1 marks)**

(c) Suppose we have marked 10 points in a 3×3 square. Use the Pigeonhole Principle to show that there are two marked points such that the distance between them is less than or equal to $\sqrt{2}$. **(4 marks)**

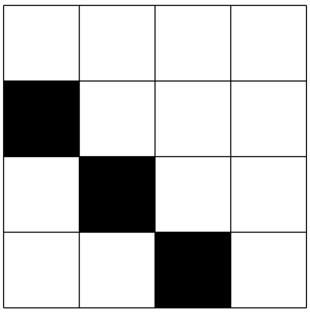
Solution:

(a) Suppose we have a set B with $|B| = n$, and subsets $A_1, \dots, A_m \subseteq B$ with $B = A_1 \cup \dots \cup A_m$. Suppose also that $m < n$; then there exists i such that $|A_i| > 1$. [2] **Bookwork. Full marks will be given for any correct statement of the same general type.**

(b) The maximum possible distance is the distance between two opposite corners, which is $r\sqrt{2}$. [1]

(c) **Unseen.** Let B be the set of marked points. We can divide the square into nine squares of size 1×1, say $Q_1, \dots, Q_9$, and put $A_i = B \cap Q_i$ [2]. By the pigeonhole principle, there exists i such that $|A_i| > 1$ [1]. We can now choose two distinct marked points $x, y \in A_i$. As these both lie in the square Q_i of side 1, we see that the distance from x to y is at most $\sqrt{2}$ [1].

(7) Consider the following board B:



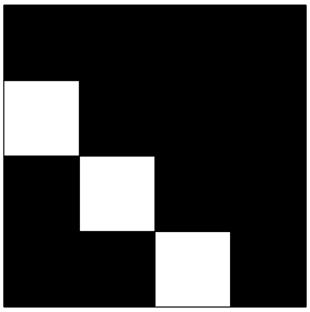
(a) Calculate the rook polynomial of the complementary board $\overline{B}$. **(2 marks)**

(b) Use (a) to calculate the number of ways of placing 4 non-challenging rooks on B. **(4 marks)**

(c) Using (b), draw a board B' such that there are precisely 121 ways to place 8 non-challenging rooks on B'. **(2 marks)**

Solution:

(a) **Standard.** The complementary board $\overline{B}$ is as follows: [1]



This is just the fully disjoint union of three 1×1 boards, so the rook polynomial is $(1+x)^3 = 1 + 3x + 3x^2 + 1$. [1]

(b) **Standard.** We use the standard relation

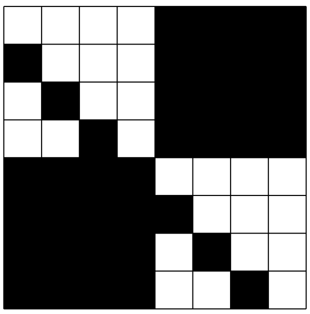
$$c_n(B) = \sum_{k=0}^{n} (-1)^k (n-k)! c_k(\overline{B}).[2]$$

This gives

$$c_4(B) = 4!c_0(\overline{B}) - 3!c_1(\overline{B}) + 2!c_2(\overline{B}) - 1!c_3(\overline{B}) + 0!c_4(\overline{B}) = 24 \times 1 - 6 \times 3 + 2 \times 3 - 1 \times 1 + 1 \times 0 = 11.[2]$$

Two marks will be given for calculating $c_4(B)$ by any other correct method.

(c) We now take B' to be the fully disjoint union of two copies of B:



This has $c_8(B') = c_4(B)^2 = 11^2 = 121$. [2]

This would be a standard calculation if B' was given. It will take some insight to work in the opposite direction.

(8)

(a) State Landau's theorem on scores in tournaments. **(4 marks)**

(b) Suppose we have a tournament with $n > 0$ players, in which every player has the same score. Show that n must be odd. **(3 marks)**

(c) Give an example of a tournament of 5 players in which every player has the same score. **(3 marks)**

(d) Use Landau's criterion to show that there is a tournament of 6 players in which every player scores 2 or 3. **(4 marks)**

(e) Find an explicit example of a tournament as in (d). **(4 marks)**

Solution:

(a) **Bookwork.** Landau's theorem is as follows. Consider a list $s_1, \dots, s_n$ of nonnegative integers with $\sum_i s_i = \binom{n}{2}$ [1]. Then the following conditions are equivalent:

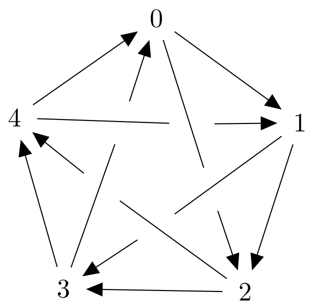
  (1) There is an n-player tournament in which player i wins s_i games for all i [1]

  (2) The sum of any k of the terms s_i is at least $\binom{k}{2}$ [1]

  (3) The sum of any k of the terms s_i is at most $\binom{k}{2} + k(n-k)$. [1]

(b) **One similar problem has been seen.** Suppose we have a tournament of $n > 0$ players in which the score of each player is k. The sum of all scores is then nk, and this must be equal to $\binom{n}{2} = n(n-1)/2$, so $k = (n-1)/2$, so $n = 2k+1$, so n is odd. [3]

(c) **Seen.** We can use the standard odd modular tournament: the set of players is $\mathbb{Z}/5$, and player i beats player j iff $j - i \in \{1, 2\}$. [3]



Full credit will be given for a correct example constructed by any means.

(d) **Standard, apart from the first step.** We have 6 scores, each of which is 2 or 3. The sum of the scores must be $\binom{6}{2} = 15$. This is only possible if we have three 3's and three 2's, so the score sequence is $3, 3, 3, 2, 2, 2$ [2]. We can check Landau's criterion as follows:

$$\begin{aligned}
2 &= 2 && \geq 0 = \tbinom{1}{2} \\
2+2 &= 4 && \geq 1 = \tbinom{2}{2} \\
2+2+2 &= 6 && \geq 3 = \tbinom{3}{2} \\
3+2+2+2 &= 9 && \geq 6 = \tbinom{4}{2} \\
3+3+2+2+2 &= 12 && \geq 10 = \tbinom{5}{2} \\
3+3+3+2+2+2 &= 15 && = 15 = \tbinom{6}{2}
\end{aligned}$$

All the required inequalities are satisfied, so Landau's Theorem tells us that there exists a tournament with this score sequence. [2]

(e) **Similar problems have been seen.** One possible solution is as follows: [4]

	0	1	2	3	4	5
0		W	W	W	L	L
1	L		W	W	W	L
2	L	L		W	W	W
3	L	L	L		W	W
4	W	L	L	L		W
5	W	W	L	L	L	

This can be seen as an adjustment of the odd modular tournament. The set of players is $\mathbb{Z}/6$. Player i beats player j if $j - i \in \{1, 2\}$, and player i loses to player j if $j - i \in \{-1, -2\} = \{4, 5\}$. This just leaves the case where $j - i = 3$, for which we declare that 0 beats 3 and 1 beats 4 and 2 beats 5. Alternatively, the table can easily be contructed by trial and error. **Full credit will be given for a correct example constructed by any means.**

(9) Find numbers $a, \dots, q$ such that the following matrix becomes a latin square: **(6 marks)**

$$\begin{bmatrix} 0 & 1 & \mathbf{a} & \mathbf{f} & \mathbf{g} \\ \mathbf{d} & \mathbf{b} & 4 & 0 & \mathbf{c} \\ \mathbf{l} & \mathbf{m} & \mathbf{n} & \mathbf{h} & \mathbf{k} \\ \mathbf{i} & 3 & 2 & \mathbf{e} & \mathbf{j} \\ \mathbf{o} & \mathbf{p} & \mathbf{q} & 1 & 3 \end{bmatrix}$$

Explain your reasoning for at least three of the entries.

Solution: Similar problems have been seen.

The completed square is as follows: [3]

$$\begin{bmatrix} 0 & 1 & 3 & 2 & 4 \\ 3 & 2 & 4 & 0 & 1 \\ 4 & 0 & 1 & 3 & 2 \\ 1 & 3 & 2 & 4 & 0 \\ 2 & 4 & 0 & 1 & 3 \end{bmatrix}$$

The following table summarises the calculation:

entry	row	column	value
a	$0, 1$	$2, 4$	3
b	$0, 4$	$1, 3$	2
c	$0, b = 2, 4$	3	1
d	$0, c = 1, b = 2, 4$	0	3
e	$2, 3$	$0, 1$	4
f	$0, 1, a = 3$	$0, 1, e = 4$	2
g	$0, 1, f = 2, a = 3$	$c = 1, 3$	4
h		$0, 1, f = 2, e = 4$	3
i	$2, 3, e = 4$	$0, d = 3$	1
j	$i = 1, 2, 3, e = 4$	$c = 1, 3, g = 4$	0
k	$h = 3$	$j = 0, c = 1, 3, g = 4$	2
l	$k = 2, h = 3$	$0, i = 1, d = 3$	4
m	$k = 2, h = 3, l = 4$	$1, b = 2, 3$	0
n	$m = 0, k = 2, h = 3, l = 4$	$2, a = 3, 4$	1
o	$1, 3$	$0, i = 1, d = 3, l = 4$	2
p	$1, o = 2, 3$	$m = 0, 1, b = 2, 3$	4
q	$1, o = 2, 3, p = 4$	$n = 1, 2, a = 3, 4$	0

For the first step, entry **a** appears in the same row as 0 and 1, and in the same column as 2 and 4, so we must have **a** $= 3$ to avoid a clash. This is recorded in the first row of the table. The second row shows that **b** $= 2$ in the same way. The third row is similar except that we now know that **b** $= 2$, and **b** appears in the same row as **c**, so **c** $\neq 2$. Proceeding in the same way, we can find **a** to **q** in alphabetical order. [3] **Full marks can be given for any coherent explanation.**

(10) Recall that if there is a block design with parameters (v, b, r, k, λ) then the following equations are satisfied:

$$(A) \quad bk = vr \qquad\qquad (B) \quad r(k-1) = \lambda(v-1)$$

(a) Prove equation (A). **(4 marks)**

(b) Suppose that $k = 2\lambda + 1$ and $b = 2k + 1$. Prove that $v = b$ and $r = k$. [Hint: write $r = k + s$, rewrite everything in terms of v, λ and s, then use (B) to find v, then use (A).] **(8 marks)**

(c) Suppose that $b = v = 7$ and the first six column sets are as follows:

$$C_1 = \{2,4,6\} \qquad C_2 = \{1,4,5\} \qquad C_3 = \{3,4,7\}$$
$$C_4 = \{1,2,3\} \qquad C_5 = \{2,5,7\} \qquad C_6 = \{1,6,7\}$$

Find the parameters r, k and λ, the corresponding row sets R_i, and the last column set C_7. **(8 marks)**

Solution:

(a) **Bookwork.** Put

$$\begin{aligned} X &= \{(j,p) \in B \times V \mid p \in C_j\} \\ &= \{(j,p) \in B \times V \mid v \in R_p\}.[2] \end{aligned}$$

We can use the first description to find $|X|$: there are b ways to choose $j \in B$, and then $|C_j| = k$ ways to choose $p \in C_j$, so $|X| = bk$ [1]. Alternatively, we can use the second description. There are v ways to choose $p \in V$, and then $|R_p| = r$ ways to choose $j \in R_p$, so $|X| = vr$ [1]. By comparing these, we see that $bk = vr$.

(b) **Unseen.** Suppose that $k = 2\lambda+1$ and $b = 2k+1 = 4\lambda+3$ and $r = k+s = 2\lambda+1+s$. Equations (A) and (B) become $(4\lambda+3)(2\lambda+1) = v(2\lambda+1+s)$ and $2\lambda(2\lambda+1+s) = \lambda(v-1)$ [2]. From (B) we get $v = 4\lambda+3+2s$, and substituting this into (A) gives $(4\lambda+3)(2\lambda+1) = (4\lambda+3+2s)(2\lambda+1+s)$ [1]. By expanding and rearranging we get $s(8\lambda+5+2s) = 0$ [2]. The solutions are $s = 0$ and $s = -4\lambda - 5/2$, but λ and s are integers so the second solution is impossible. We therefore have $s = 0$ and so $r = k = 2\lambda+1$ [2]. We have also seen that $v = 2r+1$, so $v = 4\lambda+3 = b$ [1].

(c) **Unseen.** All the sets C_j have size k, so $k = 3$ [1]. As $b = v = 7$ and $k = 3$, the relation $bk = vr$ gives $r = 3$ [1]. The relation $r(k-1) = \lambda(v-1)$ now becomes $6 = 6\lambda$, so $\lambda = 1$ [1]. Now put

$$R'_i = \{j \in \{1,\dots,6\} \mid i \in C_j\}.$$

These can be tabulated as follows:

$$R'_1 = \{2,4,6\} \qquad R'_2 = \{1,4,5\} \qquad R'_3 = \{3,4\}$$
$$R'_4 = \{1,2,3\} \qquad R'_5 = \{2,5\} \qquad R'_6 = \{1,6\}$$
$$R'_7 = \{3,5,6\}[2]$$

The row set R_i is either R'_i (if $i \notin C_7$) or $R'_i \cup \{7\}$ (if $i \in C_7$). On the other hand, each set R_i must have size $k = 3$. This can only be consistent if $C_7 = \{3,5,6\}$ [2] and

$$R_1 = \{2,4,6\} \qquad R_2 = \{1,4,5\} \qquad R_3 = \{3,4,7\}$$
$$R_4 = \{1,2,3\} \qquad R_5 = \{2,5,7\} \qquad R_6 = \{1,6,7\}$$
$$R_7 = \{3,5,6\}[1]$$