Combinatorics

(1)

(a) State the definition of the binomial coefficient $\binom{n}{k}$ in terms of factorials. **(1 marks)**

(b) State Pascal's relation for binomial coefficients. **(2 marks)**

(c) Suppose that $n \geq k \geq 2$. By thinking about the maximum and minimum elements of subsets of $\{1, \dots, n\}$, prove that **(8 marks)**

$$\binom{n}{k} = \sum_{i=1}^{n-k+1} \sum_{j=i+k-1}^{n} \binom{j-i-1}{k-2}.$$

Solution:

(a) $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ [1]

(b) For $0 < k \leq n$ we have $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. [2] (**Conditions on n and k are not required.**)

(c) Recall that $\binom{n}{k}$ is the number of subsets $A \subseteq \{1, \dots, n\}$ with $|A| = k$ [1]. To choose A, we can first choose the smallest element $i \in A$, then the largest element $j \in A$, then $k-2$ extra elements lying strictly between i and k [3]. Because there needs to be space for $k-1$ elements lying above i, the range of possible values for i is from 1 to $n-k+1$ [1]. Because there needs to be space for $k-2$ elements lying strictly between i and j, the range of possible values for j is from $i+k-1$ to n [1]. Once we have chosen i and j, we note that there are $j-i-1$ numbers strictly between them, so the number of ways of choosing the $k-2$ extra elements is $\binom{j-i-1}{k-2}$ [1]. This gives

$$\binom{n}{k} = \sum_{i=1}^{n-k+1} \sum_{j=i+k-1}^{n} \binom{j-i-1}{k-2}.$$

as claimed. [1]

Comments: Part (c) is just a slightly more complicated version of Example 16 in the notes for Chapter 1. It was disappointing that not many students succeeded with this.

(2)

(a) Suppose that $m, k \geq 0$. How many solutions are there for $u_1 + \cdots + u_k = m$ with $u_1, \dots, u_k$ being nonnegative integers? **(2 marks)**

(b) How many solutions are there for the equation $\sum_{i=1}^{9} x_i = 48$, where each x_i is a natural number with $x_i \geq i$? **(7 marks)**

Solution:

(a) The number of solutions is $\binom{m+k-1}{k-1}$. [2] **Bookwork**

(b) If we put $x_i = i + y_i$ [1], then the constraints become $y_i \geq 0$ [1], and the equation becomes

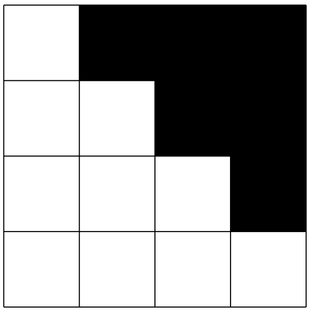
$$48 = \sum_{i=1}^{9}(i + y_i) = \sum_{i=1}^{9} i + \sum_{i=1}^{9} y_i = \frac{9 \times 10}{2} + \sum_{i=1}^{9} y_i = 45 + \sum_{i=1}^{9} y_i,$$

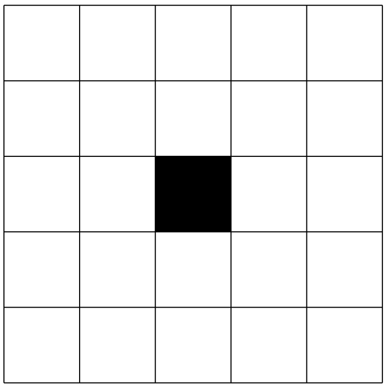
so $\sum_{i=1}^{9} y_i = 3$ [2]. By taking $k = 9$ and $m = 3$ in (a), we see that the number of solutions is

$$\binom{11}{8} [2] = \frac{11 \times 10 \times 9}{3 \times 2 \times 1} = 165. [1]$$

Comments: Most students did this well.

(3) Consider the following two boards:

(a) 

(b) 

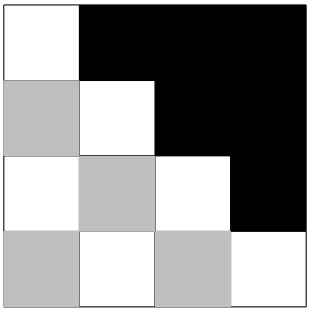
(The black squares do not count as part of the board.)

(a) Can board (a) be covered by non-overlapping dominos? Justify your answer. **(3 marks)**

(b) Can board (b) be covered by non-overlapping dominos? Justify your answer. **(3 marks)**

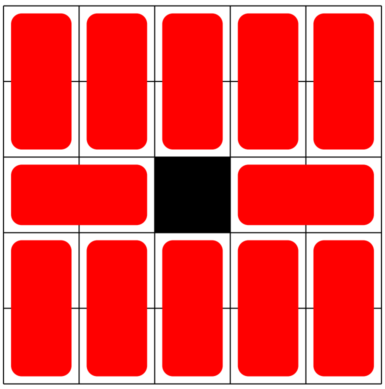
Solution:

(a) If we colour the squares in board (a) in the standard pattern, we get the following picture:

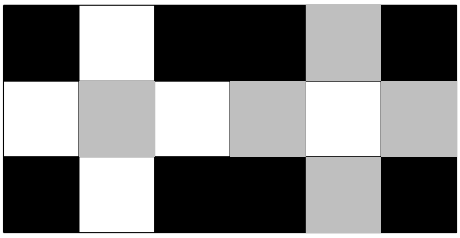


There are four grey squares and six white ones, but any set of disjoint dominos will cover the same number of grey and white squares, so we cannot cover the whole board. [3]

(b) Board (b) can be covered as follows (or in many other ways): [3]



Comments: Most students did this well. To get full marks for part (b), you needed to specify (with words or a picture) how to lay out the dominos, it is not enough to just say that the number of white and grey squares is the same. You should take note of the following example:



The number of white squares and grey squares is the same, but it is not hard to check that this shape cannot be covered by disjoint dominos.

(4)

(a) State the inclusion-exclusion principle. **(3 marks)**

(b) Let B be the set of permutations σ of $\{1,\dots,6\}$ such that either $\{\sigma(1),\sigma(2)\}=\{1,2\}$, or $\{\sigma(3),\sigma(4)\}=\{3,4\}$, or $\{\sigma(5),\sigma(6)\}=\{5,6\}$. Use the inclusion-exclusion principle to find $|B|$. **(10 marks)**

Solution:

(a) Consider a finite set B, with a list of subsets $B_1,\dots,B_n\subseteq B$. For $I\subseteq\{1,\dots,n\}$ put $B_I=\bigcap_{i\in I}B_i$, with the convention that $B_\emptyset=B$. The IEP says that

$$|B_1\cup\dots\cup B_n|=\sum_{I\neq\emptyset}(-1)^{|I|-1}|B_I|,\ \textbf{[3]}$$

or equivalently

$$|B\setminus(B_1\cup\dots\cup B_n)|=\sum_I(-1)^{|I|}|B_I|.$$

Full marks will be given for either of the equivalent forms. Versions with ellipses instead of summation notation will be accepted if they are sufficiently clear.

(b)
- Let B_1 be the set of permutations that preserve $\{1,2\}$
- Let B_2 be the set of permutations that preserve $\{3,4\}$
- Let B_3 be the set of permutations that preserve $\{5,6\}$.

Then $B=B_1\cup B_2\cup B_3$ **[1]**, so the IEP gives

$$|B|=|B_1|+|B_2|+|B_3|-|B_1\cap B_2|-|B_1\cap B_3|-|B_2\cap B_3|+|B_1\cap B_2\cap B_3|.\textbf{[1]}$$

Now, an element of B_1 can either exchange 1 and 2 or not, and can permute the set $\{3,4,5,6\}$ in any of $4!=24$ ways, so $|B_1|=2\times 4!=48$ **[2]**. Similarly, we have $|B_2|=|B_3|=48$ **[1]**.

Now consider a permutation $\sigma\in B_1\cap B_2$. This can either exchange 1 and 2 or not, and it can either exchange 3 and 4 or not, and it can permute the remaining set $\{5,6\}$ in any of $2!=2$ possible ways, which again just means that it can either exchange 5 and 6 or not. From this we see that $|B_1\cap B_2|=2\times 2\times 2=8$ **[2]**. We also see in the same way that the sets $B_1\cap B_3$, $B_2\cap B_3$ and $B_1\cap B_2\cap B_3$ are just the same as $B_1\cap B_2$ and so they all have size 8 **[2]**. Thus, the IEP becomes

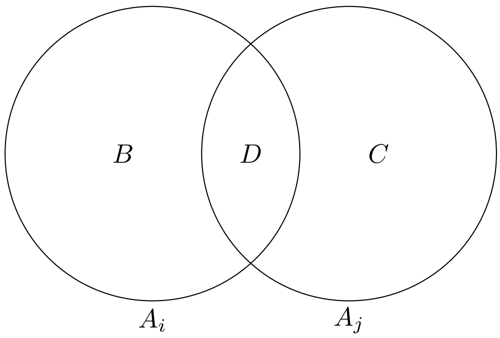
$$|B|=48+48+48-8-8-8+8=128.\textbf{[1]}$$

Questions about derangements are very standard. This is along the same lines, but a bit harder and less familiar.

Comments: In part (b), very many studenst said that if $\{\sigma(1),\sigma(2)\}=\{1,2\}$ then we must have $\sigma(1)=1$ and $\sigma(2)=2$. This is not correct: we could instead have $\sigma(1)=2$ and $\sigma(2)=1$. Most students who made this mistake obtained an answer of $|B|=67$ and were given a score of $6/10$.

(5) Let A be a subset of $\mathbb{Z}$ with $|A|=10$. Show that there are disjoint subsets $B,C\subseteq A$ with $\sum B=\sum C \pmod{1000}$. **(9 marks)**

Solution: The number of subsets of A is $2^{10} = 1024$ [2]. Let the subsets be $A_1, \dots, A_{1024}$, and let s_i be the sum of A_i modulo 1000, so s_i lies in the set $X = \{0, \dots, 999\}$, which has $|X| = 1000$ [2]. As we have 1024 elements of a set of size 1000, they cannot all be different. We can therefore find $i \neq j$ with $s_i = s_j$ [2], so $\sum A_i = \sum A_j \pmod{1000}$. The sets A_i and A_j are different, but they need not be disjoint. However, we can put $B = A_i \setminus A_j$ and $C = A_j \setminus A_i$ and $D = A_i \cap A_j$ [1], so we have the following Venn diagram:



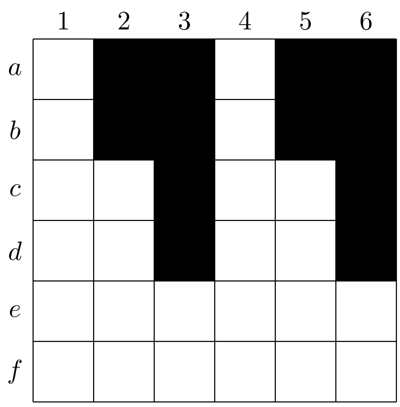
Then B and C are disjoint, and we have

$$\sum B + \sum D = \sum A_i = \sum A_j = \sum C + \sum D \pmod{1000},$$

so $\sum B = \sum C \pmod{1000}$ [2].

Comments: This combines some ideas from Example 28 (in the notes for Chapter 2) with some ideas from Example 29. Many students gave a partially correct answer, but very few were completely correct.

(6) How many ways are there of placing six non-challenging rooks on the following board? **(7 marks)**



Hint: you do not need to use any theorems; it is easier to argue directly.

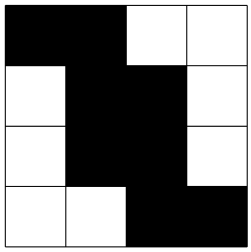
Solution: We need a rook in each row. For rows a and b, the only possible choices are $a1b4$ and $a4b1$ [2]. Either way, this uses columns 1 and 4, leaving only columns 2 and 5 for the rooks in rows c and d. Thus, the only possibilities for these rows are $c2d5$ and $c5d2$ [2]. Either way, this uses columns 2 and 5, leaving only columns 3 and 6 for the rooks in rows e and f. Thus, the only possibilities for these rows are $e3f6$ and $e6f3$ [1]. From this we see that the total number of possibilities is $2 \times 2 \times 2 = 8$ [2].

The analysis can also be written in tabular form as follows:

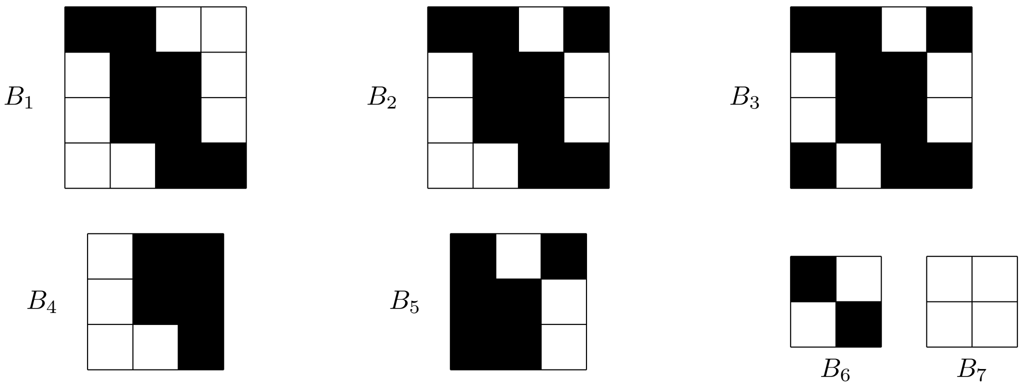
$a1$	$\underline{b4}$	$c2$	$\underline{d5}$	$e3$	$\underline{f6}$	✓
				$\underline{e6}$	$\underline{f3}$	✓
		$\underline{c5}$	$\underline{d2}$	$e3$	$\underline{f6}$	✓
				$\underline{e6}$	$\underline{f3}$	✓
$\underline{a4}$	$\underline{b1}$	$c2$	$\underline{d5}$	$e3$	$\underline{f6}$	✓
				$\underline{e6}$	$\underline{f3}$	✓
		$\underline{c5}$	$\underline{d2}$	$e3$	$\underline{f6}$	✓
				$\underline{e6}$	$\underline{f3}$	✓

Comments: Most students did this well.

(7) Calculate the rook polynomial for the following board: **(13 marks)**



Solution: We consider boards as follows:



Let $p_k(x)$ be the rook polynomial for B_k, so our task is to calculate $p_1(x)$. By inspection, we have

$$p_4(x) = 1 + 4x + 2x^2 \text{[1]}$$
$$p_5(x) = 1 + 3x + 2x^2 \text{[1]}$$
$$p_6(x) = 1 + 2x + x^2 \text{[1]}$$
$$p_7(x) = 1 + 4x + 2x^2. \text{[1]}$$

Board $B3$ splits as the union of B_6 (in rows $1, 4$ and columns $2, 3$) with B_7 (in rows $2, 3$ and columns $1, 4$) [2], so we have

$$p_3(x) = p_6(x)p_7(x) = 1 + 6x + 11x^2 + 8x^3 + 2x^4. \text{[1]}$$

Next, B_3 is obtained from B_2 by filling in the bottom left square, whereas B_5 is obtained from B_2 by deleting the row and column of the bottom left square, so we have

$$p_2(x) = p_3(x) + x\,p_5(x) = 1 + 7x + 14x^2 + 10x^3 + 2x^4. \text{[3]}$$

Similarly, B_2 and B_5 are obtained from B_1 using the top right square, so

$$p_1(x) = p_2(x) + x\,p_4(x) = 1 + 8x + 18x^2 + 12x^3 + 2x^4.\textbf{[3]}$$

Comments: Most students did this well, either by a variation of the method shown above, or by the tabular method (for which full marks were also given). By far the most common mistake was to use the relation $r_B(x) = r_C(x)r_D(x)$ in cases where it is not valid. This relation only works if none of the squares in C are in the same row or column as any of the squares in D.

(8)

(a) State Landau's theorem on scores in tournaments. **(4 marks)**

(b) Now consider a tournament of 8 players.

(1) Can there be three players who each win at least six matches? If you think that this is possible, then give an example of such a tournament; if not, give a proof of impossibility.

(2) Can there be four players who each win at least six matches? If you think that this is possible, then give an example of such a tournament; if not, give a proof of impossibility.

(8 marks)

Solution:

(a) Landau's theorem is as follows. Consider a list $s_1, \dots, s_n$ of nonnegative integers with $\sum_i s_i = \binom{n}{2}$ **[1]**. Then the following conditions are equivalent:

(1) There is an n-player tournament in which player i wins s_i games for all i **[1]**

(2) The sum of any k of the terms s_i is at least $\binom{k}{2}$ **[1]**

(3) The sum of any k of the terms s_i is at most $\binom{k}{2} + k(n-k)$. **[1]**

(b) (1) Suppose that in every match, the lower-numbered player wins, except in the match between players 1 and 3, in which case player 3 wins. Then

* Player 1 beats $2, 4, 5, 6, 7, 8$
* Player 2 beats $3, 4, 5, 6, 7, 8$
* Player 3 beats $1, 4, 5, 6, 7, 8$
* Player 4 beats $5, 6, 7, 8$
* Player 5 beats $6, 7, 8$
* Player 6 beats $7, 8$
* Player 7 beats 8
* Player 8 beats no one.

Thus, the scores are $6, 6, 6, 4, 3, 2, 1, 0$, so in particular there are three players who win at least six matches. **[4]**

One can also show that the above pattern of scores is realisable using Landau's numerical criterion:

$$0 \geq \binom{1}{2} = 0 \quad \checkmark \qquad 1 + 0 \geq \binom{2}{2} = 1 \quad \checkmark$$

$$2 + 1 + 0 \geq \binom{3}{2} = 3 \quad \checkmark \qquad 3 + 2 + 1 + 0 \geq \binom{4}{2} = 6 \quad \checkmark$$

$$4 + 3 + 2 + 1 + 0 \geq \binom{5}{2} = 10 \quad \checkmark \qquad 6 + 4 + 3 + 2 + 1 + 0 \geq \binom{6}{2} = 15 \quad \checkmark$$

$$6 + 6 + 4 + 3 + 2 + 1 + 0 \geq \binom{7}{2} = 21 \quad \checkmark \qquad 6 + 6 + 6 + 4 + 3 + 2 + 1 + = \binom{8}{2} = 28 \quad \checkmark.$$

However, this only earns partial credit, because the question asks for an example. (Maximum of 3 marks for this approach.)

(2) The upper version of Landau's criterion says that in a tournament of n players, the sum of any k scores is at most $\binom{k}{2} + k(n-k)$ [2]. Taking $n = 8$ and $k = 4$, we see that the sum of any four scores is at most $\binom{4}{2} + 4 \times 4 = 22$. (This number can also be obtained as $\sum_{i=1}^{k}(n-i) = 7+6+5+4$.) [1]If there were four players who won at least six matches each, then the sum of their scores would be at least 24, which is impossible by the above criterion. [1]

Comments: Most students did not give a clear and careful formulation of Landau's theorem, and so scored only about 2/4 for part (a). Many students did not give a completely specified example for (b)(2), and so scored only 2/4 or 3/4.

(9) Find numbers $a, \ldots, x$ such that the following matrix becomes a latin square: **(8 marks)**

$$\left[\begin{array}{cccc|cc} \mathbf{a} & 3 & 2 & 4 & \mathbf{b} & \mathbf{c} \\ 2 & 4 & 6 & 5 & \mathbf{d} & \mathbf{e} \\ 4 & 6 & \mathbf{f} & 3 & \mathbf{g} & \mathbf{h} \\ 6 & 5 & 4 & 2 & \mathbf{i} & \mathbf{j} \\ \hline \mathbf{k} & \mathbf{m} & \mathbf{n} & \mathbf{p} & \mathbf{q} & \mathbf{r} \\ \mathbf{s} & \mathbf{t} & \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{x} \end{array}\right]$$

Solution: Full marks will be given for finding a correct matrix by any means.

First consider the top left 4×4 block. We need to fill in a and f to make a 4×4 latin square that is extendable to a 6×6 latin square. [1]By the standard criterion, this means that the number of occurrences of each entry in the 4×4 block should be at least $4 + 4 - 6 = 2$ [1]. The numbers of occurrences of $2, \ldots, 6$ are $3, 2, 4, 2, 3$, but 1 does not occur at all. Thus, the only way to satisfy the criterion is to take $a = f = 1$. [1]

This gives a 4×4 latin square, in which 1, 3 and 5 occur twice, and everything else occurs at least three times. We can draw this square, together with the options for columns 5 and 6, as follows:

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 5/6 & 6/5 \\ 2 & 4 & 6 & 5 & 1/3 & 3/1 \\ 4 & 6 & 1 & 3 & 2/5 & 5/2 \\ 6 & 5 & 4 & 2 & 1/3 & 3/1 \end{bmatrix} \text{[1]}$$

We need to fill in column 5 in such a way that the extendibility criterion is still satisfied. This means that we need one extra occurrence of 1, 3 and 5, so these numbers must appear somewhere in column 5 [1]. Once we have filled in column 5, there will be a unique way to fill in column 6. There are four possibilities for columns 5 and 6, as follows:

$$\begin{bmatrix} 5 & 6 \\ 1 & 3 \\ 2 & 5 \\ 3 & 1 \end{bmatrix} \text{[1]} \qquad \begin{bmatrix} 5 & 6 \\ 3 & 1 \\ 2 & 5 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} 6 & 5 \\ 1 & 3 \\ 5 & 2 \\ 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 6 & 5 \\ 3 & 1 \\ 5 & 2 \\ 1 & 3 \end{bmatrix}$$

It is sufficient to find any one of these, which is easily done by trial and error. For the rest of this solution, we will just use the first possibility.

We now have a 4×6 latin rectangle, which we need to extend to 6×6. The options for the last two rows can be displayed as follows:

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 1 & 3 \\ 4 & 6 & 1 & 3 & 2 & 5 \\ 6 & 5 & 4 & 2 & 3 & 1 \\ 3/5 & 1/2 & 3/5 & 1/6 & 4/6 & 2/4 \\ 5/3 & 2/1 & 5/3 & 6/1 & 6/4 & 4/2 \end{bmatrix} \text{[1]}$$

It turns out that no backtracking is needed here, we can just fill in row 5 by using the first option that has not already been used, giving $(3, 1, 5, 6, 4, 2)$, and then row 6 is forced to be $(5, 2, 3, 1, 6, 4)$. We end up with the following latin square:

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 1 & 3 \\ 4 & 6 & 1 & 3 & 2 & 5 \\ 6 & 5 & 4 & 2 & 3 & 1 \\ 3 & 1 & 5 & 6 & 4 & 2 \\ 5 & 2 & 3 & 1 & 6 & 4 \end{bmatrix} \text{[1]}$$

Comments: Most students did this correctly.

(10)

(a) Define what it means for two latin squares to be orthogonal. **(3 marks)**

(b) It is given that the following matrices are latin squares, and that they are all orthogonal to each other:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \\ e & f & g & h \\ i & j & k & m \end{bmatrix}$$

(1) Using the orthogonality of A and C, show that none of b, g and m can be equal to 1. Find similar restrictions for the rest of the variables. **(3 marks)**

(2) Use the orthogonality of B and C in the same way to find nine more restrictions. **(2 marks)**

(3) Use the fact that C is a latin square to give nine more restrictions. **(2 marks)**

(4) Find all the entries in C. **(2 marks)**

Solution:

(a) Consider two $n \times n$ latin squares M and N, and let P be the $n \times n$ matrix whose (i, j)'th entry is the ordered pair (M_{ij}, N_{ij}). We say that M and N are *orthogonal* if every pair (p, q) (with $1 \leq p, q \leq n$) appears precisely once in P. [3]

(b) (1) The orthogonality condition for A and C says that the following matrix should contain each pair pq precisely once:

$$\begin{bmatrix} 11 & 22 & 33 & 44 \\ 2a & 1b & 4c & 3d \\ 3e & 4f & 1g & 2h \\ 4i & 3j & 2k & 1m \end{bmatrix}$$

The pair 11 appears on the top row, so it cannot appear again anywhere else, so none of the pairs $1b$, $1g$ and $1m$ can be equal to 11, so none of b, g or m can be equal to 1. [2] By the same logic, none of a, h and k can be equal to 2, and none of d, e and j can be equal to 3, and none of c, f and i can be equal to 4. In summary, no letter can be equal to the number with which it is paired, so

$$\begin{array}{cccc} a \neq 2 & b \neq 1 & c \neq 4 & d \neq 3 \\ e \neq 3 & f \neq 4 & g \neq 1 & h \neq 2 \\ i \neq 4 & j \neq 3 & k \neq 2 & m \neq 1. \end{array}$$ [1]

(2) Again, no letter can be equal to the number in the corresponding place in B, so

$$\begin{array}{cccc} a \neq 3 & b \neq 4 & c \neq 1 & d \neq 2 \\ e \neq 4 & f \neq 3 & g \neq 2 & h \neq 1 \\ i \neq 2 & j \neq 1 & k \neq 4 & m \neq 3. \end{array}$$ [2]

(3) Because C is a latin square, no letter can be equal to the number at the top of the same column, so

$$\begin{array}{cccc} a \neq 1 & b \neq 2 & c \neq 3 & d \neq 4 \\ e \neq 1 & f \neq 2 & g \neq 3 & h \neq 4 \\ i \neq 1 & j \neq 2 & k \neq 3 & m \neq 4. \end{array}$$ [2]

(4) The above restrictions leave only one possibility for each number, namely

$$\begin{array}{cccc} a = 4 & b = 3 & c = 2 & d = 1 \\ e = 2 & f = 1 & g = 4 & h = 3 \\ i = 3 & j = 4 & k = 1 & m = 2. \end{array}$$

Thus

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{bmatrix} \ \textbf{[2]}$$

Comments: Most students did this correctly, although many lost one or two marks by not giving a clear and careful statement in part (a).