The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

Combinatorics **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (a) State the definition of the binomial coefficient $\binom{n}{k}$ in terms of factorials.

(1 mark)

(b) State Pascal's relation for binomial coefficients. ***(2 marks)***

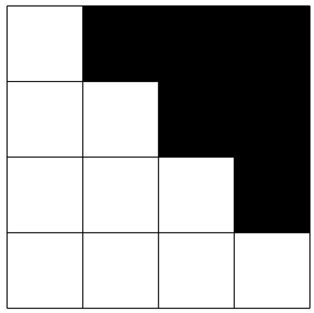
(c) Suppose that $n \geq k \geq 2$. By thinking about the maximum and minimum elements of subsets of $\{1, \ldots, n\}$, prove that ***(8 marks)***

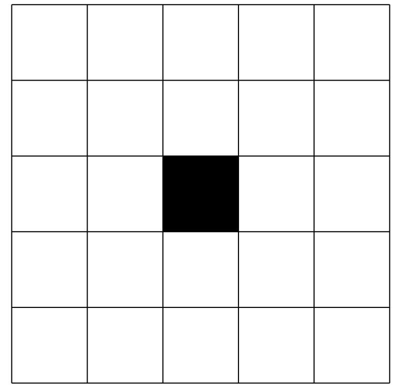
$$\binom{n}{k} = \sum_{i=1}^{n-k+1} \sum_{j=i+k-1}^{n} \binom{j-i-1}{k-2}.$$

2 (a) Suppose that $m, k \geq 0$. How many solutions are there for $u_1 + \cdots + u_k = m$ with $u_1, \ldots, u_k$ being nonnegative integers? ***(2 marks)***

(b) How many solutions are there for the equation $\sum_{i=1}^{9} x_i = 48$, where each x_i is a natural number with $x_i \geq i$? ***(7 marks)***

3 Consider the following two boards:

(a) 

(b) 

(The black squares do not count as part of the board.)

(a) Can board (a) be covered by non-overlapping dominos? Justify your answer. ***(3 marks)***

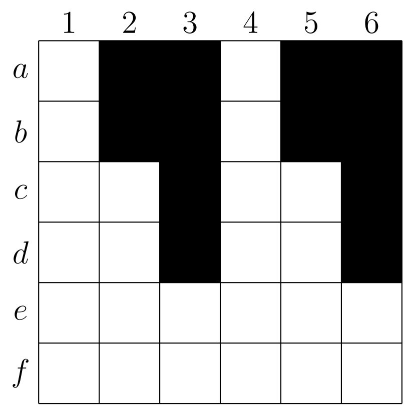
(b) Can board (b) be covered by non-overlapping dominos? Justify your answer. ***(3 marks)***

4 (a) State the inclusion-exclusion principle. ***(3 marks)***

(b) Let B be the set of permutations σ of $\{1, \ldots, 6\}$ such that either $\{\sigma(1), \sigma(2)\} = \{1, 2\}$, or $\{\sigma(3), \sigma(4)\} = \{3, 4\}$, or $\{\sigma(5), \sigma(6)\} = \{5, 6\}$. Use the inclusion-exclusion principle to find $|B|$. ***(10 marks)***

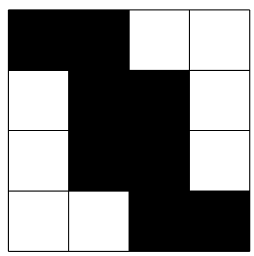
5 Let A be a subset of $\mathbb{Z}$ with $|A| = 10$. Show that there are disjoint subsets $B, C \subseteq A$ with $\sum B = \sum C \pmod{1000}$. ***(9 marks)***

6 How many ways are there of placing six non-challenging rooks on the following board? ***(7 marks)***



Hint: you do not need to use any theorems; it is easier to argue directly.

7 Calculate the rook polynomial for the following board: ***(13 marks)***



8 (a) State Landau's theorem on scores in tournaments. ***(4 marks)***

(b) Now consider a tournament of 8 players.

(1) Can there be three players who each win at least six matches? If you think that this is possible, then give an example of such a tournament; if not, give a proof of impossibility.

(2) Can there be four players who each win at least six matches? If you think that this is possible, then give an example of such a tournament; if not, give a proof of impossibility.

(8 marks)

9 Find numbers $a, \ldots, x$ such that the following matrix becomes a latin square: ***(8 marks)***

$$\left[\begin{array}{cccc|cc} \mathbf{a} & 3 & 2 & 4 & \mathbf{b} & \mathbf{c} \\ 2 & 4 & 6 & 5 & \mathbf{d} & \mathbf{e} \\ 4 & 6 & \mathbf{f} & 3 & \mathbf{g} & \mathbf{h} \\ 6 & 5 & 4 & 2 & \mathbf{i} & \mathbf{j} \\ \hline \mathbf{k} & \mathbf{m} & \mathbf{n} & \mathbf{p} & \mathbf{q} & \mathbf{r} \\ \mathbf{s} & \mathbf{t} & \mathbf{u} & \mathbf{v} & \mathbf{w} & \mathbf{x} \end{array}\right]$$

10 (a) Define what it means for two latin squares to be orthogonal. ***(3 marks)***

(b) It is given that the following matrices are latin squares, and that they are all orthogonal to each other:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \\ 2 & 1 & 4 & 3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \\ e & f & g & h \\ i & j & k & m \end{bmatrix}$$

(1) Using the orthogonality of A and C, show that none of b, g and m can be equal to 1. Find similar restrictions for the rest of the variables. ***(3 marks)***

(2) Use the orthogonality of B and C in the same way to find nine more restrictions. ***(2 marks)***

(3) Use the fact that C is a latin square to give nine more restrictions. ***(2 marks)***

(4) Find all the entries in C. ***(2 marks)***

End of Question Paper