MAS334 COMBINATORICS — AUTUMN SEMESTER 2015-2016
EXAM SOLUTIONS AND MARK SCHEME

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Solution to Question 1

ia) **(standard)** For $n \in \mathbb{N}$, by the Binomial Theorem,

$$(1-x)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i. \qquad (\textbf{1 Mark})$$

So

$$\begin{aligned} 1-(1-x)^n &= 1 - \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i = 1 - 1 + \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} x^i \\ &= \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} x^i \qquad (\textbf{1 Mark}) \end{aligned}$$

and so

$$\frac{1-(1-x)^n}{x} = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} x^{i-1}. \qquad (\textbf{1 Mark})$$

ib) **(unseen, easy but there as a hint for the next part)**

$$\frac{1-(1-x)^n}{x} = \frac{1-y^n}{1-y} = 1 + y + y^2 + \cdots + y^{n-1}. \qquad (\textbf{2 Marks})$$

ic) **(unseen)**

Integrating the identity from (a) with respect to x gives

$$\int \frac{1-(1-x)^n}{x}\, dx = \int \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} x^{i-1}\, dx. \qquad (\textbf{1 Mark})$$

$$\Rightarrow \qquad \int \sum_{k=0}^{n-1} y^k \frac{dx}{dy} dy = \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i} \binom{n}{i} x^i + c$$

$$\Rightarrow \qquad \int - \sum_{k=0}^{n-1} y^k dy = \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i} \binom{n}{i} x^i + c$$

$$\Rightarrow \qquad - \sum_{k=0}^{n-1} \frac{y^{k+1}}{k+1} = \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i} \binom{n}{i} x^i + c$$

So

$$-\sum_{k=1}^{n}\frac{(1-x)^k}{k}=\sum_{i=1}^{n}(-1)^{i-1}\frac{1}{i}\binom{n}{i}x^i+c.$$

(3 Marks)

[**1** for the integral on LHS, **1** for the integral on RHS, **1** for remembering the constant of integration]

To determine the constant of integration c, set $x=0$, giving

$$c=-\sum_{k=1}^{n}\frac{1}{k}. \qquad (\textbf{1 Mark})$$

Finally, to obtain the required identity, set $x=1$, giving

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{1}{i}\binom{n}{i}=-c=\sum_{k=1}^{n}\frac{1}{k}. \qquad (\textbf{1 Mark})$$

iia) **(bookwork)** Consider a $k-1$ by n grid. By a standard procedure involving counting x_i units of progress from left to right along the i-th horizontal grid line up, it may be seen that the required solutions are in bijection with shortest routes from bottom left to top right in the grid. Therefore there are $\binom{n+k-1}{k-1}$ such solutions. **(3 Marks)**

iib) **(bookwork)** Suppose we have a finite set of items and properties $1,2,3,\dots,n$. Let $N(i_1,i_2,\dots,i_r)$ be the number of items which have the properties $i_1,i_2,\dots,i_r$ (and maybe others). Then the number of items with at least one of the properties is

$$\begin{aligned}
&N(1)+N(2)+N(3)+\cdots+N(n)\\
&-N(1,2)-N(1,3)-\cdots-N(n-1,n)\\
&+N(1,2,3)+N(1,2,4)+\cdots\\
&-N(1,2,3,4)-\cdots\\
&\qquad\vdots\\
&+(-1)^{n-1}N(1,2,3,\dots,n).
\end{aligned}$$

(3 Marks)

iic) **(similar to homework problem)** As in the first part, the total number of non-negative integer solutions to the given equation is

$$\binom{21+4-1}{3}=\binom{24}{3}=2024.$$

Let P_1 be the property that $x_1\geqslant 7$, P_2 the property that $x_2\geqslant 9$ and P_3 the property that $x_3\geqslant 12$. We want the number of non-negative integer solutions with *none* of the properties P_1, P_2, P_3. **(2 Marks)**

So this is

total no. of solutions – no. of solutions with at least one of the properties

$$= \binom{24}{3} - \left(N(1) + N(2) + N(3) - N(1,2) - N(1,3) - N(2,3) + N(1,2,3)\right),$$

where we have adopted usual I/E notation. **(1 Mark)**
Now, if $x_1 \geqslant 7$, write $x_1 = 7 + y_1$ where y_1 is a non-negative integer. So non-negative integer solutions of the original equation with $x_1 \geqslant 7$ correspond to non-negative integer solutions of $y_1 + x_2 + x_3 + x_4 = 14$. So, by part (i), $N(1) = \binom{17}{3}$. **(2 Marks)**
Similarly, $N(2) = \binom{15}{3}$, $N(3) = \binom{12}{3}$, $N(1,2) = \binom{8}{3}$, and $N(1,3) = \binom{5}{3}$, $N(2,3) = \binom{3}{3}$, $N(1,2,3) = 0$. **(2 Marks)**
So the answer is

$$\binom{24}{3} - \binom{17}{3} - \binom{15}{3} - \binom{12}{3} + \binom{8}{3} + \binom{5}{3} + \binom{3}{3} - 0 \qquad (\textbf{1 Mark})$$
$$= 2024 - 680 - 455 - 220 + 56 + 10 + 1 = 736.$$

Solution to Question 2

i) **(unseen)** If we work mod 2, it makes no difference if we insert plus or minus signs. Each even number contributes zero and each of the five odd numbers contributes 1, giving us 1 mod 2. So there is no choice of signs giving 0. **(3 Marks)**

ii) **(unseen)** The number of squares is $3n - 3 = 3(n-1)$. **(1 Mark)**
If n is even, the number of squares is odd, so this is impossible. **(1 Mark)**
If $n = 2m + 1$ is odd, colour the squares alternately black and white and consider the numbers of black and white squares. **(1 Mark)**
Suppose the corners are black. Then the number of black squares is
$$\frac{3(n-1)}{2} + 1 = 3m + 1$$
and the number of white squares is $3m - 1$. **(1 Mark)**
But dominoes cover the same number of squares of each colour, so it's impossible. **(1 Mark)**

iiia) **(bookwork)** If you place more than n letters in n pigeon-holes then some pigeon-hole will contain more than one letter. **(2 Marks)**

iiib) **(unseen)**
Each person has either 0, 1, 2, 3 or 4 friends in the group. **(1 Mark)**
Suppose someone has 0 friends in the group. Notice that means that none of the group has 4 friends, because then they would be friends with everyone else. (Friendship is reciprocal.) **(1 Mark)**
So, in this case, we can label pigeon-holes 0 to 3 and place each of the five people in the PH corresponding to their number of friends in the group. **(1 Mark)**

By the PHP, there will be two in the same PH, as required. **(1 Mark)**
On the other hand, if no-one has 0 friends, we repeat the procedure with PHs 1 to 4. **(1 Mark)**

iiic) **(unseen)** Label 1789 pigeon-holes by the possible remainders on division by 1789. **(1 Mark)**
Place each of 1790 numbers of the form 1, 11, 111, ... into the pigeon-hole corresponding to its remainder on division by 1789. **(1 Mark)**
By the pigeon-hole principle, there will be two numbers in the same pigeon-hole. **(1 Mark)**
Their difference is therefore divisible by 1789. **(1 Mark)**
This difference is of the form $111...10...0 = 111...1 \cdot 10^r$. Since 1789 and 10^r are coprime, 1789 must divide 111...1. **(1 Mark)**

iva) **(unseen, easy)** For example, $7, 5, 1, 3, 8$. **(1 Mark)**

ivb) **(unseen, easy)** No. 5 can only be chosen from A_2, but then, apart from in A_2, 6 and 8 both only appear in A_5, so cannot be simultaneously picked. **(2 Marks)**

ivc) **(bookwork)** Sets $A_1, A_2, \ldots A_n$ have distinct representatives if and only if, for each r, any collection of r of the sets contains at least r elements. **(2 Marks)**

Solution to Question 3

i) **(standard problem)**
There are many ways of calculating, using Theorem 43 (select a square) and Theorem 46 (disjoint boards) from the course.
The easiest is to first use the square top left in Theorem 43, arriving at boards to which we can apply Theorem 46. **(2 Marks)**
We get:

$$r_B(x) = r_C(x) + xr_D(x),$$

where C consists of a full 2×2 board and a disjoint 2×3 board with one corner square shaded and D consists of two disjoint 1×2 boards. **(3 Marks)**

So

$$\begin{aligned} r_B(x) &= (1 + 4x + 2x^2)(1 + 5x + 4x^2) + x(1 + 2x)^2 \\ &= 1 + 10x + 30x^2 + 30x^3 + 8x^4 \,. \qquad (\textbf{3 Marks}) \end{aligned}$$

[**Alternative:** Direct counting by hand will be given full marks if the correct answer is obtained; partial marks may be obtained for a partially correct answer depending on the explanation given for the counting procedure adopted.]

ii) **(bookwork)** We apply the IEP. The items are all $n!$ ways of placing n non-challenging rooks on the full $n \times n$ board. Property i is that the rook in row i is in B. **(3 Marks)**
So $N(i_1, i_2, \ldots, i_r)$ is the number of layouts of n non-challenging rooks on an $n \times n$ board such that the rooks in rows i_1, i_2, ..., i_r are in B.

(1 Mark)

The number of ways of placing n non-challenging rooks on $\overline{B}$ is the number of the $n!$ items having none of the properties. **(1 Mark)**

This is $n!$ minus the number of items with at least one of the properties. **(1 Mark)**

By the IEP, this is

$$\begin{aligned}
&n! - N(1) - N(2) - N(3) - \cdots - N(n) \\
&+ N(1,2) + N(1,3) + \cdots + N(n-1,n) \\
&- N(1,2,3) - N(1,2,4) - \cdots \\
&+ N(1,2,3,4) + \cdots \\
&\qquad\qquad \vdots \\
&+ (-1)^n N(1,2,3,\ldots,n).
\end{aligned}$$

(2 Marks)

Now $N(i_1, i_2, \ldots, i_s)$ is the number of ways of placing s rooks in rows i_1, i_2, ..., i_s of B multiplied by $(n-s)!$, the number of ways of placing the remaining $n-s$ rooks. **(2 Marks)**

So the sum of all the $N(i_1, i_2, \ldots, i_s)$ terms is $r_s(n-s)!$. **(1 Mark)**

The required result comes from putting this into the formula above. **(1 Mark)**

iiia) **(standard problem)** By parts (i) and (ii), the required number is

$$120 - 24.10 + 6.30 - 2.30 + 1.8 - 1.0 = 120 - 240 + 180 - 60 + 8 = 8.$$

(2 Marks)

[Follow through from the candidate's answer to (i).]

iiib) **(unseen, but link between permutations and rooks was seen in an example)**

Permutations correspond to placement of non-challenging rooks on a full board, and conditions on permutations are enforced by shading squares.

We see that this is the number of ways of placing 5 non-challenging rooks on $\overline{B}$, so the answer is 8 by part a). **(3 Marks)**

[A calculation directly by hand gets the marks, if the link to the rest of the question is missed.]

Solution to Question 4

i) **(standard problem)** We adopt standard Latin square notation. Here $p = 3$, $q = 4$ and $n = 6$. The extension is possible iff $L(i) \geq 3 + 4 - 6 = 1$ for $1 \leq i \leq 6$. This happens iff $x = 5$. **(1 Mark)**

One extension is

$$\begin{pmatrix} 3 & 1 & 2 & 4 & 5 & 6 \\ 1 & 3 & 6 & 2 & 4 & 5 \\ 4 & 6 & 5 & 3 & 1 & 2 \\ 2 & 4 & 1 & 5 & 6 & 3 \\ 5 & 2 & 4 & 6 & 3 & 1 \\ 6 & 5 & 3 & 1 & 2 & 4 \end{pmatrix} \qquad (\textbf{6 Marks})$$

[Marking: 4 marks for any correct extension to 3×6 or 6×4; 2 marks for any correct extension from there to 6×6. Partial marks available for "partially correct" extensions.]

ii) **(unseen)**

iia) Either set up a suitable tournament, with players $p_1, \ldots, p_n$, such that p_i beats each p_j with $j > i$. Thus p_i scores $n - i$, as required.
Alternatively, check that the conditions of Landau's Theorem are satisfied, by noting that for each r the sum of the smallest r scores is $1 + 2 + \cdots + (r-1) = \binom{r}{2}$. **(3 Marks)**

iib) We exhibit a suitable tournament. It has three subtournaments A, B, C each as in part a). All players from subtournament A beat all those from B (and lose to those from C). All players from subtournament B beat all those from C (and lose to those from A). So all players from subtournament C beat all those from A (and lose to those from B). Each player therefore has their score from their subtournament plus n, resulting in the required list of scores. **(4 Marks)**
[Note: This could also be done with Landau's Theorem, but it's a bit messy, so I didn't put "Hence, or otherwise" in the question.]

iii) **(unseen)**

iiia) A given number appears in the block corresponding to itself and the block of each of the other nine numbers that do not share a row or column with it, hence it appears in 10 blocks. **(2 Marks)**

iiib) Let x, y be a pair of numbers.
First consider the case where x and y are in the same row. They appear together in exactly 6 blocks, namely those corresponding to squares not in their row and not in either of their columns. **(2 Marks)**
Similarly if x and y are in the same column. **(1 Mark)**
Now suppose x and y are in different rows and different columns. Then they appear together in exactly 6 blocks, namely the block of square x, the block of square y and the four blocks of squares sharing neither row nor column with x or y. **(3 Marks)**
iiic) Clearly, we have 16 varieties, the numbers $1, 2, \ldots, 16$. And we have 16 blocks, one for each square. **(1 Mark)**
Each block contains 10 varieties and each variety appears in 10 blocks. **(1 Mark)**
Each pair of varieties appears in precisely 6 blocks.

Thus we have a $(16, 16, 10, 10, 6)$ design. **(1 Mark)**