The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2015–16**

Combinatorics **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) (a) Let $n \geq 1$. Use the Binomial Theorem to show that

$$\frac{1-(1-x)^n}{x} = \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} x^{i-1}.$$

(3 marks)

(b) Show that

$$\frac{1-(1-x)^n}{x} = 1 + y + \cdots + y^{n-1},$$

where $y = 1 - x$. ***(2 marks)***

(c) By integrating the expression in part (a), show that

$$\sum_{i=1}^{n} \frac{1}{i} = \sum_{i=1}^{n} (-1)^{i-1} \frac{1}{i} \binom{n}{i}.$$

(6 marks)

(ii) (a) How many solutions are there of the equation

$$x_1 + x_2 + \cdots + x_k = n,$$

in which each x_i is a non-negative integer? Give a brief reason for your answer. ***(3 marks)***

(b) State the Inclusion/Exclusion Principle. ***(3 marks)***

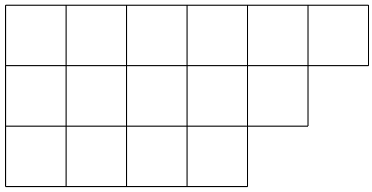
(c) How many solutions are there of the equation

$$x_1 + x_2 + x_3 + x_4 = 21,$$

in which each x_i is a non-negative integer, such that $x_1 < 7$, $x_2 < 9$ and $x_3 < 12$? ***(8 marks)***

2 (i) The numbers 1 to 10 are written in a row. Can one insert plus and minus signs between them in such a way that the value of the resulting expression is zero? ***(3 marks)***

(ii) Consider a $3 \times n$ rectangle with the three squares in one corner removed. (The case $n = 6$ is pictured below.)



Show that this cannot be completely covered by non-overlapping dominoes (that is, by pieces which cover exactly two adjacent squares). ***(5 marks)***

(iii) (a) State the Pigeon-hole Principle. ***(2 marks)***

(b) Show that in any group of five people, there are two who have the same number of friends within the group. ***(5 marks)***

(c) Show that there exists an integer whose decimal representation consists entirely of 1s (that is, an integer of the form 111...1) which is divisible by 1789. ***(5 marks)***

(iv) (a) Find distinct representatives of the sets

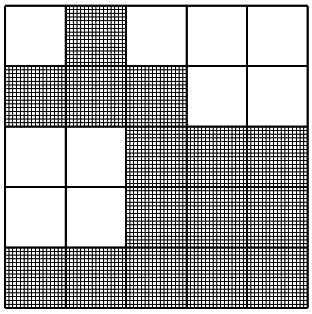
$$A_1 = \{1, 2, 7\},$$
$$A_2 = \{5, 6, 8\},$$
$$A_3 = \{1, 3, 7\},$$
$$A_4 = \{2, 3, 4, 7\},$$
$$A_5 = \{1, 2, 6, 8\}.$$

(1 mark)

(b) Can distinct representatives of these sets be chosen to include 5, 6 and 8? ***(2 marks)***

(c) State a necessary and sufficient condition for sets A_1, A_2, ..., A_n to have distinct representatives. ***(2 marks)***

3 (i) Calculate the rook polynomial of the (unshaded) board B:



(8 marks)

(ii) Let B be part of an $n \times n$ board with rook polynomial

$$1 + r_1 x + r_2 x^2 + \cdots + r_n x^n$$

and let $\overline{B}$ be the complement of B. Prove that the number of ways of placing n non-challenging rooks on $\overline{B}$ is

$$\sum_{k=0}^{n} (-1)^k (n-k)! r_k,$$

where $r_0 = 1$. ***(12 marks)***

(iii) (a) Calculate the coefficient of x^5 in the rook polynomial of $\overline{B}$, where B is the board in part (i). ***(2 marks)***

(b) Using a relationship between permutations and non-challenging rooks, or otherwise, find the number of permutations of $\{1, 2, 3, 4, 5\}$ satisfying the following conditions.

$$1 \mapsto 2, 2 \not\mapsto 4, 2 \not\mapsto 5, 3 \not\mapsto 1, 3 \not\mapsto 2, 4 \not\mapsto 1, 4 \not\mapsto 2.$$

(3 marks)

4 (i) For what value of x can the following Latin rectangle be extended to a 6×6 Latin square?

$$\begin{pmatrix} 3 & 1 & 2 & 4 \\ 1 & 3 & 6 & 2 \\ 4 & 6 & x & 3 \end{pmatrix}$$

Write down one such extension. ***(7 marks)***

(ii) (a) Show that there is a tournament of n players with scores

$$(n-1, n-2, n-3, \ldots, 2, 1, 0).$$

(3 marks)

(b) Hence show that there is a tournament of $3n$ players with scores

$$(2n-1, 2n-1, 2n-1, 2n-2, 2n-2, 2n-2, \ldots, n+1, n+1, n+1, n, n, n).$$

(4 marks)

(iii) Consider a 4×4 board with squares labelled by the numbers $1, 2, \ldots, 16$ as shown.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Define blocks as follows. For each square on the board, form a block consisting of the number on that square together with the nine numbers not sharing a row or column with that square. For example, $\{1, 6, 7, 8, 10, 11, 12, 14, 15, 16\}$ is a block, corresponding to the top left square.

(a) Show that each number is in 10 blocks. ***(2 marks)***

(b) Show that each pair of numbers appears in precisely 6 blocks. ***(6 marks)***

(c) Deduce that the blocks make up a $(16, 16, 10, 10, 6)$ design. ***(3 marks)***

End of Question Paper