THE CONSTRUCTION OF SINGULAR (CO)HOMOLOGY

N. P. STRICKLAND

Contents

1. Cochain complexes and homological algebra

We will construct cohomology rings for spaces in two stages. First, for each space X we construct a graded ring $C^*(X)$ with an additional structure called a differential. Next, for each differential graded ring we define an associated cohomology ring. This section develops the general theory of differential graded rings, and various related structures, that are needed for the second stage.

Definition 1.1. A *cochain complex* is a system of abelian groups C^k (for $k \in \mathbb{Z}$) equipped with homomorphisms $d_C^k \colon C^k \to C^{k+1}$ (called *differentials*) such that the composites

$$C^{k-1} \xrightarrow{d_C^{k-1}} C^k \xrightarrow{d_C^k} C^{k+1}$$

are all zero. In most contexts we will just write d^k or d instead of d_C^k, so the above condition is just $dd = 0$. We write C^* for the whole system of groups. We say that C^* is *nonnegative* if $C^k = 0$ for all $k < 0$. (This will be the usual case for us.)

Definition 1.2. A *differential graded ring* is a cochain complex C^* with a compatible ring structure. In more detail, there should be a product rule giving an element $ab \in C^{i+j}$ for each $a \in C^i$ and $b \in C^j$, and this should satisfy $(ab)c = a(bc)$ and $(a + a')(b + b') = ab + ab' + a'b + a'b'$. There should also be a specified unit element $1 \in C^0$ such that $1a = a = a1$ for all a. These structures should interact with the differential by the rules $d(1) = 0$ and $d(ab) = d(a)b + (-1)^i a\, d(b)$ (for $a \in C^i$ and $b \in C^j$ as before). This last identity is called the *Leibniz rule*.

Remark 1.3. Note that we do *not* assume a commutativity rule $ab = ba$ or $ab = (-1)^{ij} ba$. Later we will show how to define a cohomology ring $H^*(C^*)$ associated to C^*. In the cases of interest, this cohomology ring will be graded-commutative (so $ab = (-1)^{ij} ba$) even though C^* itself is not. This is actually an important phenomenon, related to the existence of Steenrod operations in the cohomology of spaces, but we will not explore that here.

Example 1.4. Any graded ring can be considered as a differential graded ring, just by taking the differential d to be the zero map. Note that if C^* is a differential graded ring with $C^{2i+1} = 0$ for all i, then $d \colon C^k \to C^{k+1}$ is automatically zero for all k, because either k or $k + 1$ must be odd.

Date: January 11, 2023.

Example 1.5. Consider the polynomial ring $A^* = \mathbb{Z}[a,x]/a^2$, which has a basis over $\mathbb{Z}$ consisting of the monomials x^i and $x^i a$ (for $i \geq 0$). We can make this a graded ring by giving a degree one and x degree two, so $A^{2i} = \mathbb{Z}.x^i$ and $A^{2i+1} = \mathbb{Z}.x^i a$ for $i \geq 0$. We can then define a differential $d\colon A^k \to A^{k+1}$ by $d(x^i) = 0$ and $d(x^i a) = x^{i+1}$. We claim that this gives a differential graded ring. The only thing that needs checking is the Leibniz rule. There are four cases to consider:

$$\begin{aligned}
d(x^i x^j) &= d(x^{i+j}) = 0 & d(x^i)x^j + x^i d(x^j) &= 0.x^j + x^i.0 = 0\\
d(ax^i x^j) &= d(ax^{i+j}) = x^{i+j+1} & d(ax^i)x^j - ax^i d(x^j) &= x^{i+1}.x^j - ax^i.0 = x^{i+j+1}\\
d(x^i ax^j) &= d(ax^{i+j}) = x^{i+j+1} & d(x^i)ax^j + x^i d(ax^j) &= 0.ax^j + x^i.x^{j+1} = x^{i+j+1}\\
d(ax^i ax^j) &= d(0) = 0 & d(ax^i)ax^j - ax^i d(ax^j) &= x^{i+1}.ax^j - ax^i.x^{j+1} = 0.
\end{aligned}$$

Example 1.6. For another similar example, we can consider $B^* = \mathbb{Z}[y,b]/b^2$, graded so that $y \in B^2$ and $b \in B^3$. Then for $i \geq 0$ we have $B^{2i} = \mathbb{Z}.y^i$ and $B^{2i+1} = \mathbb{Z}.y^{i-1}b$ except that $B^1 = 0$. We now define $d\colon B^k \to B^{k+1}$ by $d(y^i) = iy^{i-1}b$ and $d(y^i b) = 0$. One can check (by separating four cases as before) that the Leibniz formula is satisfied, so we have a differential graded ring.

Note that the differential here is actually completely determined by the fact that $d(y) = b$. Given this, a straightforward induction with the Leibniz formula shows that $d(y^k)$ must be $ky^{k-1}b$. Also, the rule $dd = 0$ shows that $d(b) = dd(y) = 0$, and we can use Leibniz again (with the fact that $b^2 = 0$) to see that $d(y^k b) = 0$ as well.

Example 1.7. Readers familiar with differential geometry may wish to consider the following example. Let M be a smooth manifold, and let $\Omega^k(M)$ denote the ring of smooth differential k-forms over M. The usual exterior product of differential forms and the de Rham differential $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$ make $\Omega^*(M)$ into a differential graded ring.

Definition 1.8. Let C^* be a cochain complex. We put

$$\begin{aligned}
Z^k(C^*) &= \ker(d_C^k\colon C^k \to C^{k+1})\\
B^k(C^*) &= \mathrm{image}(d_C^{k-1}\colon C^{k-1} \to C^k)\\
H^k(C^*) &= Z^k(C^*)/B^k(C^*).
\end{aligned}$$

(For the definition of $H^k(C^*)$ to make sense, we need to know that $B_k(C^*) \leq Z_k(C^*)$. This follows immediately from the condition $d^2 = 0$.) The elements of $Z^k(C^*)$ are called *cocycles*, and the elements of $B^k(C^*)$ are called *coboundaries*. The groups $H^k(C^*)$ are the *cohomology groups* of the complex, and the elements are called *cohomology classes*. When C^* is clear from the context, we will just write Z^k for $Z^k(C^*)$ and so on.

Proposition 1.9. *Suppose that C^* is a differential graded ring. Then there is a well-defined product on $H^*(C^*)$ given by*

$$(a + B^i)(b + B^j) = ab + B^{i+j}$$

for all $a \in Z^i$ and $b \in Z^j$. This makes $H^*(C^*)$ into a graded ring.

Proof. Any other representative for the coset $a + B^i$ has the form $a' = a + du$ for some $u \in C^{i-1}$. Similarly, any other representative for the coset $b + B^i$ has the form $b' = b + dv$ for some $v \in C^{j-1}$. Note also that a and b are elements of Z^*, so $da = 0$ and $db = 0$. We then find that

$$a'b' = ab + (du)b + a(dv) + (du)(dv) = ab + d(ub + (-1)^i av + u\,dv),$$

which represents the same coset as ab. It follows that we have a well-defined product. This is associative because

$$\begin{aligned}
((a + B^i)(b + B^j))(c + B^k) &= (ab + B^{i+j})(c + B^k) = (ab)c + B^{i+j+k}\\
&= a(bc) + B^{i+j+k} = (a + B^i)(bc + B^{j+k})\\
&= (a + B^i)((b + B^j)(c + B^k)).
\end{aligned}$$

All the other ring axioms for $H^*(C^*)$ follow in a similar way from the corresponding axioms for C^*. $\square$

Example 1.10. For the differential graded ring A^* in example 1.5, we have $Z^*(A^*) = \mathbb{Z}[x]$ and $B^*(A^*) = \mathbb{Z}[x].x$ so $H^*(A^*) = \mathbb{Z}[x]/x = \mathbb{Z}$. More explicitly, we have $H^0(A^*) = \mathbb{Z}$, and $H^i(A^*) = 0$ for all $i \neq 0$.

Example 1.11. For the differential graded ring B^* in Example 1.6, the subgroup of cocycles is generated by 1 together with the elements $y^i b$ for $i \geq 0$, whereas the coboundaries are generated by the elements $(i+1)y^i b$. In particular, the element b is a coboundary, but $y^i b$ is not when $i > 0$. Now let u_i be the cohomology class of $y^i b$ (for $i > 0$). We find that $H^{2i+3}(B^*)$ is a copy of $\mathbb{Z}/(i+1)$ generated by u_i, and that $H^0(A^*) = \mathbb{Z}$, and all the other cohomology groups are trivial. The groups $H^*(A^*)$ form a graded ring, but the ring structure is not very interesting: as $b^2 = 0$ we find that $u_i u_j = 0$ for all $i, j > 0$.

Example 1.12. For the cochain complex $\Omega^*(M)$ in Example 1.7, the cohomology ring $H^*(\Omega^*(M))$ is known as the *de Rham cohomology ring* of M. If M is a compact manifold of dimension d, then it is known that $H^i(\Omega^*(M))$ is a finite-dimensional vector space over $\mathbb{R}$, which is zero for $i < 0$ or $i > d$.

Definition 1.13. Let A^* and B^* be cochain complexes. A *cochain map* from A^* to B^* is a sequence of maps $\phi^k : A^k \to B^k$ for all $k \in \mathbb{Z}$ such that the following diagrams commute:

$$\begin{tikzcd} A^k \arrow[r, "\phi^k"] \arrow[d, "d_A^k"'] & B^k \arrow[d, "d_B^k"] \\ A^{k+1} \arrow[r, "\phi^{k+1}"'] & B^{k+1} \end{tikzcd}$$

Where there is no danger of confusion, we will suppress all subscripts and superscripts, so the commutation condition is just $d\phi = \phi d$. A *homomorphism of differential graded rings* is just a cochain map that is also a ring homomorphism.

Example 1.14. Let A^* be as in Examples 1.5. For any $n \in \mathbb{Z}$, we can define a DGR homomorphism $\phi_m : A^* \to A^*$ by $\phi_m(a) = ma$ and $\phi_m(x) = mx$ (so $\phi_m(x^i) = m^i x^i$ and $\phi_m(x^i a) = m^{i+1} x^i a$). We claim that these are the only such DGR homomorphisms. To see this, let $\psi : A^* \to A^*$ be an arbitrary DGR homomorphism. As $a \in A^1$ we must have $\psi(a) \in A^1 = \mathbb{Z}a$, so $\psi(a) = ma$ for some $m \in \mathbb{Z}$. We then have $\psi(x) = \psi(da) = d\psi(a) = d(ma) = mx$, and then $\psi(x^i) = \psi(x)^i = (mx)^i = m^i x^i$ and similarly $\psi(x^i a) = m^{i+1} x^i a$, so $\psi = \phi_m$ as claimed.

Now let B^* be as in Example 1.6, and suppose we have a DGR homomorphism $\zeta : A^* \to B^*$. This must have $\zeta(1) = 1$, but we claim that $\zeta(A^i) = 0$ for all $i > 0$. To see this, we first note that $\zeta(a) \in B^1 = 0$, so $\zeta(a) = 0$. As ζ is a cochain map this means that $\zeta(x) = \zeta(da) = d\zeta(a) = d(0) = 0$. As ζ is a ring homomorphism it follows in turn that that $\zeta(x^i) = 0^i = 0$ for all $i > 0$, and similarly $\zeta(x^i a) = \zeta(x^i)\zeta(a) = 0$ for $i \geq 0$, so ζ vanishes on all elements in degree greater than 0.

Example 1.15. Let $f : M \to N$ be a smooth map between smooth manifolds. Note that $\Omega^0(N)$ is just the ring of smooth real-valued functions on N, so there is an obvious ring map $f^* : \Omega^0(N) \to \Omega^0(M)$ given by $f^*(u) = u \circ f : M \to \mathbb{R}$. One could hope to extend this to give a homomorphism $f^* : \Omega^*(N) \to \Omega^*(M)$ of DGRs. For a form $\alpha = u \, dv_1 \wedge \cdots \wedge dv_r \in \Omega^r(N)$, any extension would have to satisfy

$$f^*(\alpha) = (u \circ f) \, d(v_1 \circ f) \wedge \cdots \wedge d(v_r \circ f).$$

It follows easily from this that there is at most one extension, and with more work it can be shown that there is precisely one extension.

Remark 1.16. Let $\phi : U^* \to V^*$ be a cochain map between cochain complexes. We then have graded abelian groups $\ker(\phi)$, $\mathrm{image}(\phi)$ and $\mathrm{cok}(\phi)$ given by

$$\begin{aligned} \ker(\phi)^k &= \ker(\phi^k : U^k \to V^k) = \{a \in U^k \mid \phi^k(a) = 0\} \\ \mathrm{image}(\phi)^k &= \mathrm{image}(\phi^k : U^k \to V^k) = \{b \in V^k \mid b = \phi^k(a) \text{ for some } a \in U^k\} \\ \mathrm{cok}(\phi)^k &= \mathrm{cok}(\phi^k : U^k \to V^k) = V^k / \mathrm{image}(\phi)^k. \end{aligned}$$

If $a \in \ker(\phi)^k$ then $\phi(a) = 0$ so $\phi(d(a)) = d(\phi(a)) = 0$ so $d(a) \in \ker(\phi)^{k-1}$. It follows that the differential $d : U^k \to U^{k-1}$ restricts to give a differential $d : \ker(\phi)^k \to \ker(\phi)^{k-1}$, making $\ker(\phi)$ into another cochain complex. By similar arguments, image(ϕ) and cok(ϕ) can both be considered as cochain complexes in a natural way.

Proposition 1.17. *Let $\phi\colon U^* \to V^*$ be a cochain map. Then there are well-defined maps*

$$\phi_*\colon H^k(U^*) \to H^k(V^*)$$

given by

$$\phi_*(a + B^k(U^*)) = \phi(a) + B^k(V^*)$$

for all $a \in Z^k(U^*)$. Moreover, these are functorial: if we have another cochain map $\psi\colon V^* \to W^*$ then

$$(\psi\phi)_* = \psi_* \circ \phi_*\colon H^k(U) \to H^k(W),$$

and also $(1_U)_* = 1_{H^k(U)}$. Further, if U^* and V^* are differential graded rings, and ϕ is a homomorphism of DGRs, then ϕ_* is a homomorphism of graded rings.

Proof. First, if $a \in Z^k(U^*)$ then $d\phi(a) = \phi(da) = \phi(0) = 0$, so $\phi(a) \in Z^k(V^*)$, so the expression $\phi(a) + B^k(V^*)$ defines an element of $H^k(V^*)$. Any other representative of the coset $a + B^k(V^*)$ has the form $a' = a + du$ for some $u \in U^{k-1}$. We then have $\phi(a') = \phi(a) + \phi(du) = \phi(a) + d\phi(u)$, which represents the same coset as $\phi(a)$. It follows that we have a well-defined map ϕ_* given by $\phi_*(a+B^k(U^*)) = \phi(a)+B^k(V^*)$, and this is easily seen to be a homomorphism. If we have another cochain map $g\colon V^* \to W^*$ then clearly

$$(\psi\phi)_*(a + B^k(U^*)) = \psi(\phi(a)) + B^k(W^*) = (\psi_* \circ \phi_*)(a + B^k(U^*)),$$

which proves the functorality property. Similarly, if U^* and V^* have ring structures and ϕ preserves them, we see that

$$\begin{aligned}
\phi_*((a + B^k(U^*))(b + B^l(U^*))) &= \phi_*(ab + B^{k+l}(U^*)) = \phi(ab) + B^{k+l}(V^*) \\
&= \phi(a)\phi(b) + B^{k+l}(V^*) = (\phi(a) + B^k(V^*))(\phi(b) + B^l(V^*)) \\
&= \phi_*(a + B^k(U^*))\ \phi_*(b + B^l(U^*)),
\end{aligned}$$

which proves that ϕ_* is a ring homomorphism. □

We next need to discuss a criterion that allows us to prove that different cochain maps have the same effect in cohomology.

Definition 1.18. Let $\phi, \psi\colon U^* \to V^*$ be cochain maps between cochain complexes. A *cochain homotopy* from ϕ to ψ is a system of maps $\sigma^k\colon U^k \to V^{k-1}$ such that $\psi^k - \phi^k = \sigma^{k+1}d_U^k + d_V^{k-1}\sigma^k$ (or more briefly, $\psi - \phi = d\sigma + \sigma d$). The maps in question here can be displayed in the following diagram, which is not commutative:

$$\begin{tikzcd}
U^k \arrow[r, "d_U^k"] \arrow[d, "\sigma^k"'] \arrow[dr, "\psi^k-\phi^k"] & U^{k+1} \arrow[d, "\sigma^{k+1}"] \\
V^{k-1} \arrow[r, "d_V^{k-1}"'] & V^k.
\end{tikzcd}$$

We say that ϕ and ψ are *cochain homotopic* (written $\phi \simeq \psi$) if there exists a cochain homotopy between them.

Remark 1.19. The zero maps $U^k \to V^{k+1}$ give a cochain homotopy from ϕ to itself. If σ is a cochain homotopy from ϕ to ψ, and τ is a cochain homotopy from ψ to χ, then $\sigma + \tau$ is a cochain homotopy from ϕ to χ, and $-\sigma$ is a cochain homotopy from ψ to ϕ. It follows that cochain homotopy defines an equivalence relation on the set of all cochain maps from U^* to V^*.

Now suppose we have further cochain maps $T^* \xrightarrow{\mu} U^*$ and $V^* \xrightarrow{\nu} W^*$, as well as a cochain homotopy σ from ϕ to ψ. It is then easy to see that the maps

$$\lambda^k = \nu^{k-1} \circ \sigma^k \circ \mu^k\colon T^k \to W^{k-1}$$

give a cochain homotopy from $\nu\phi\mu$ to $\nu\psi\mu$. This implies that our equivalence relation is compatible with all kinds of composition.

Proposition 1.20. *If $\phi, \psi\colon U^* \to V^*$ are chain homotopic, then $H^*(\phi) = H^*(\psi)\colon H^*(U^*) \to H^*(V^*)$.*

Proof. Choose a chain homotopy σ, so $\psi = \phi + d\sigma + \sigma d$. Consider an element $\alpha = a + B^k(U^*) \in H^k(U^*)$, so $a \in U^k$ with $d(a) = 0$. It follows that $\psi(a) = \phi(a) + d\sigma(a) + \sigma d(a) = \phi(a) + d\sigma(a) \in \phi(a) + B^k(V^*)$, so $\psi(a)$ and $\phi(a)$ represent the same cohomology class, so $\phi_*(\alpha) = \psi_*(\alpha)$ as required. $\square$

We next need some constructions that give rise to long exact sequences of cohomology groups. We first recall the basic definition.

Definition 1.21. Consider a sequence $A_0 \xrightarrow{f_0} A_1 \to \cdots \xrightarrow{f_{r-1}} A_r$ of abelian groups and homomorphisms. We say that the sequence is *exact* at A_i if $\text{image}(f_{i-1}) = \ker(f_i) \leq A_i$ (which implies that $f_i \circ f_{i-1} = 0$). We say that the whole sequence is exact if it is exact at A_i for $0 < i < r$.

Next, we say that a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is *short exact* if it is exact, and also f is injective and g is surjective.

Now suppose instead that we have a sequence $A^* \xrightarrow{\phi} B^* \xrightarrow{\psi} C^*$ of cochain complexes and cochain maps. We will say that the sequence is short exact iff for each k, the corresponding sequence $A^k \xrightarrow{\phi^k} B^k \xrightarrow{\psi^k} C^k$ of abelian groups is short exact.

Remark 1.22. One can easily check the following facts.

(a) A sequence $A \xrightarrow{f} B \xrightarrow{0} C$ is exact iff f is surjective. In particular, a sequence $A \xrightarrow{f} B \to 0$ is exact iff f is surjective.

(b) A sequence $A \xrightarrow{0} B \xrightarrow{g} C$ is exact iff g is injective. In particular, a sequence $0 \to B \xrightarrow{g} C$ is exact iff g is injective.

(c) A sequence $A \xrightarrow{0} B \xrightarrow{g} C \xrightarrow{0} D$ is exact iff g is an isomorphism.

(d) A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact iff $A \xrightarrow{f} B \xrightarrow{g} C$ is short exact.

(e) Suppose we have an exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} E.$$

Then g induces a map from $\text{cok}(f) = B/f(A)$ to C, and h can be regarded as a map from C to $\ker(k)$, and the resulting sequence

$$\text{cok}(f) \xrightarrow{g} C \xrightarrow{h} \ker(k)$$

is short exact.

(f) If $A \xrightarrow{f} B \xrightarrow{g} C$ is short exact, then f induces an isomorphism $A \to f(A)$ and g induces an isomorphism $B/f(A) \to C$. Thus, if A, B and C are finite we have $|B| = |f(A)|.|B/f(A)| = |A||C|$. Similarly, if A and C are free abelian groups of ranks n and m, then B is a free abelian group of rank $n+m$.

Proposition 1.23 (The five lemma)**.** *Suppose we have a commutative diagram as follows, in which the rows are exact, and p_0, p_1, p_3 and p_4 are isomorphisms:*

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, "p_0"', "\simeq"] & A_1 \arrow[r, "f_1"] \arrow[d, "p_1"', "\simeq"] & A_2 \arrow[r, "f_2"] \arrow[d, "p_2"'] & A_3 \arrow[r, "f_3"] \arrow[d, "p_3"', "\simeq"] & A_4 \arrow[d, "p_4"', "\simeq"] \\
B_0 \arrow[r, "g_0"'] & B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3 \arrow[r, "g_3"'] & B_4.
\end{tikzcd}$$

Then p_2 is also an isomorphism.

Proof. First suppose that $a_2 \in A_2$ and $p_2(a_2) = 0$. It follows that $p_3 f_2(a_2) = g_2 p_2(a_2) = g_2(0) = 0$, but p_3 is an isomorphism, so $f_2(a_2) = 0$, so $a_2 \in \ker(f_2)$. The top row is exact, so $\ker(f_2) = \text{image}(f_1)$, so we can choose $a_1 \in A_1$ with $f_1(a_1) = a_2$. Put $b_1 = p_1(a_1) \in B_1$. We then have $g_1(b_1) = g_1 p_1(a_1) = p_2 f_1(a_1) = p_2(a_2) = 0$, so $b_1 \in \ker(g_1)$. The bottom row is exact, so $\ker(g_1) = \text{image}(g_0)$, so we can choose $b_0 \in B_0$ with $g_0(b_0) = b_1$. As p_0 is an isomorphism, we can now put $a_0 = p_0^{-1}(b_0) \in A_0$. We then have $p_1 f_0(a_0) = g_0 p_0(a_0) = g_0(b_0) = b_1 = p_1(a_1)$. Here p_1 is an isomorphism, so it follows that $f_0(a_0) = a_1$. We now have $a_2 = f_1(a_1) = f_1 f_0(a_0)$. However, as the top row is exact we have $f_1 f_0 = 0$, so $a_2 = 0$. We conclude that p_2 is injective.

Now suppose instead that we start with an element $b_2 \in B_2$. Put $b_3 = g_2(b_2) \in B_3$ and $a_3 = p_3^{-1}(b_3) \in A_3$. We then have $p_4 f_3(a_3) = g_3 p_3(a_3) = g_3(b_3) = g_3 g_2(b_2) = 0$ (because $g_3 g_2 = 0$). As p_4 is an isomorphism, this means that $f_3(a_3) = 0$, so $a_3 \in \ker(f_3)$. As the top row is exact we have $\ker(f_3) = \operatorname{image}(f_2)$, so we can choose $a_2 \in A_2$ with $f_3(a_2) = a_3$. Put $b_2' = b_2 - p_2(a_2) \in B_2$. We have $g_2(b_2') = g_2(b_2) - g_2 p_2(a_2) = b_3 - p_3 f_2(a_2) = b_3 - p_3(a_3) = 0$, so $b_2' \in \ker(g_2) = \operatorname{image}(g_1)$. We can thus choose $b_1' \in B_1$ with $g_1(b_1') = b_2'$. Now put $a_1' = p_1^{-1}(b_1') \in A_1$ and $a_2' = f_1(a_1') \in A_2$. We find that $p_2(a_2') = p_2 f_1(a_1') = g_1 p_1(a_1') = g_1(b_1') = b_2' = b_2 - p_2(a_2)$, so $p_2(a_2 + a_2') = b_2$. This shows that p_2 is also surjective, and so is an isomorphism as claimed. □

Proposition 1.24. *Suppose we have a short exact sequence $U^* \xrightarrow{\phi} V^* \xrightarrow{\psi} W^*$ of cochain complexes. Then there is a natural system of maps $\delta^k \colon H^k(W^*) \to H^{k+1}(U^*)$ (called* connecting homomorphisms*) such that the sequence*

$$H^k(U^*) \xrightarrow{\phi_*} H^k(V^*) \xrightarrow{\psi_*} H^k(W^*) \xrightarrow{\delta} H^{k+1}(U^*) \xrightarrow{\phi_*} H^{k+1}(V^*)$$

is exact for all k.

For simplicity we have stated this in a form that does not include a definition of the maps δ^k, and indeed the definition is not needed for most applications. The slogan behind the definition is just that $\delta = \phi^{-1} d \psi^{-1}$, but considerable work is required to make this precise. We will need the details in order to analyse the properties of δ, so we explain them now.

Definition 1.25. In the context of Proposition 1.24, a *snake* is a list

$$\sigma = (\overline{w}, w, v, u, \overline{u}) \in H^k(W^*) \times Z^k(W^*) \times V^k \times Z^{k+1}(U^*) \times H^{k+1}(U^*)$$

such that:

(a) $\overline{w}$ is the cohomology class of w, and $\overline{u}$ is the cohomology class of u.
(b) $\psi(v) = w$, and $\phi(u) = d(v)$.

More specifically, we say here that σ is a *snake from $\overline{w}$ to $\overline{u}$*. We write S for the set of all snakes, which is clearly a group under addition.

Lemma 1.26. *For any element $\overline{w} \in H^k(W^*)$, there exists a snake starting with $\overline{w}$.*

Proof. We can certainly choose a representing cocycle $w \in Z^k(W^*)$. As $\psi \colon V^k \to W^k$ is surjective, we can choose $v \in V^k$ with $\psi(v) = w$. We now have $\psi(dv) = d\psi(v) = dw = 0$, and $\ker(\psi) = \operatorname{image}(\phi)$ by assumption, so we can find $u \in U^{k+1}$ with $\phi(u) = dv$. This means that $\phi(du) = d\phi(u) = ddv = 0$, but ϕ is injective, so $du = 0$. This means that $u \in Z^{k+1}(U^*)$, so there is an associated cohomology class $\overline{u} \in H^{k+1}(U^*)$. We now see that $(\overline{w}, w, v, u, \overline{u})$ is a snake starting with $\overline{w}$, as required. □

Lemma 1.27. *If two snakes have the same starting point, then they also have the same end point.*

Proof. By subtracting the two snakes, we reduce to the following statement: if we have a snake σ of the form $(0, w, v, u, \overline{u})$, then $\overline{u}$ is also zero. As the sequence is a snake we see that the cohomology class of w is trivial, so $w = d\tilde{w}$ for some $\tilde{w} \in W^{k-1}$. As ψ is surjective we can then choose $\tilde{v} \in V^{k-1}$ with $\psi(\tilde{v}) = \tilde{w}$. Now put $v' = v - d(\tilde{v}) \in V^k$. As $dd = 0$ we have $d(v') = d(v)$, which is the same as $\phi(u)$ by the snake conditions. We also have $\psi(v') = \psi(v) - d\psi(\tilde{v}) = \psi(v) - d\tilde{w} = w - w = 0$, so $v' \in \ker(\psi) = \operatorname{image}(\phi)$, so we can choose $u' \in U^{k-1}$ with $\phi(u') = v'$. Now $\phi(u - du') = d(v) - d(\phi(u')) = dv - dv' = 0$ and ϕ is injective so $u = du'$. This means that u is a coboundary, so the cohomology class $\overline{u}$ is zero as claimed. □

Lemmas 1.26 and 1.27 validate the following definition:

Definition 1.28. For any $\overline{w} \in H^k(W^*)$, we define $\delta(\overline{w})$ to be the end point of any snake that starts with $\overline{w}$.

Heuristically, in a snake $(\overline{w}, w, v, u, \overline{u})$ we see that v is a choice of $\psi^{-1}(w)$ and u is a choice of $\phi^{-1} dv$ so our definition is compatible with the slogan $\delta = \psi^{-1} d \phi^{-1}$. Using the fact that the snakes form a group under addition, it is easy to see that $\delta \colon H^k(W^*) \to H^{k+1}(U^*)$ is a homomorphism.

Proof of Proposition 1.24. We first check that all adjacent composites in our sequence are zero. Firstly, we have $\psi_*\phi_* = (\psi\phi)_* = 0_* = 0$, as required. Next, suppose we have a cohomology class $\overline{v} \in H^k(V^*)$, and we choose a representing cycle $v \in Z^k(V^*)$. One can then check that $(\psi_*(\overline{v}), \psi(v), v, 0, 0)$ is a snake, showing that $\delta\psi_*(\overline{v}) = 0$. This means that $\delta\psi_* = 0$, as required. Finally, consider an arbitrary snake $(\overline{w}, w, v, u, \overline{u})$. We then see that $\phi_*(\overline{u})$ is represented by the cocycle $\phi(u)$, but $\phi(u) = dv$ so that cohomology class is zero. This shows that $\phi_*\delta = 0$. As all these composites vanish, we deduce that $\mathrm{image}(\phi_*) \leq \ker(\psi_*)$, and $\mathrm{image}(\psi_*) \leq \ker(\delta)$, and $\mathrm{image}(\delta) \leq \ker(\phi_*)$.

We must now prove the opposite inclusions. First suppose we have a class $\overline{v} \in H^k(V^*)$ with $\psi_*(\overline{v}) = 0$. Choose a representing cocycle $v \in Z^k(V^*)$. Then $\psi(v)$ represents zero, so $\psi(v) = d\tilde{w}$ for some $\tilde{w} \in W^{k-1}$. As ψ is surjective, we can then choose $\tilde{v} \in V^{k-1}$ with $\psi(\tilde{v}) = \tilde{w}$. Now put $v' = v - d\tilde{v}$, which is another cocycle representing $\overline{v}$. We also have $\psi(v') = \psi(v) - d\psi(\tilde{v}) = \psi(v) - d\tilde{w} = 0$, so $v' \in \ker(\psi) = \mathrm{image}(\phi)$, so we can choose $u \in U^k$ with $\phi(u) = v'$. Now $\phi(du) = d\phi(u) = dv' = dv - dd\tilde{v} = 0$ and ϕ is injective so $du = 0$. We therefore have a corresponding cohomology class $\overline{u} \in H^k(U^*)$, and the relation $\phi(u) = v'$ implies that $\phi_*(\overline{u}) = \overline{v}$, so $\overline{v} \in \mathrm{image}(\phi_*)$. This completes the proof that $\ker(\psi_*) = \mathrm{image}(\phi_*)$.

Now suppose instead we start with a class $\overline{w} \in H^k(W^*)$ with $\delta(\overline{w}) = 0$. We can thus choose a snake of the form $\sigma = (\overline{w}, w, v, u, 0)$. This means that the cohomology class of u is trivial, so $u = d(\tilde{u})$ for some $\tilde{u} \in U^k$. Put $v' = v - \phi(\tilde{u}) \in V^k$. We find that $dv' = dv - \phi(d\tilde{u}) = dv - \phi(u)$, which is zero by the snake conditions for σ. Because $\psi\phi = 0$ we also have $\psi(v') = \psi(v) = w$. This means that v' represents a cohomology class $\overline{v}$ with $\psi_*(\overline{v}) = \overline{w}$, so $\overline{w} \in \mathrm{image}(\psi_*)$ as required.

Finally, suppose we start with a class $\overline{u} \in H^{k+1}(U^*)$ with $\phi_*(\overline{u}) = 0$. Choose a representing cocycle $u \in Z^{k+1}(U^*)$. Then $\phi(u)$ represents $\phi_*(\overline{u}) = 0$, so we must have $\phi(u) = dv$ for some $v \in V^k$. Put $w = \psi(v) \in U^k$, and note that $dw = d\psi(v) = \psi(dv) = \psi\phi(u) = 0$ (because $\psi\phi = 0$). This means that w is a cocyle, and we write $\overline{w}$ for the corresponding cohomology class. We now see that $(\overline{w}, w, v, u, \overline{u})$ is a snake, so $\overline{u} = \delta(\overline{w}) \in \mathrm{image}(\delta)$, as required. $\square$

The following result, known as the *Snake Lemma*, can be regarded as a special case of Proposition 1.24.

Proposition 1.29. *Suppose we have a commutative diagram as follows, in which the rows are short exact sequences:*

$$\begin{tikzcd} A \arrow[r, tail, "j"] \arrow[d, "f"'] & B \arrow[r, two heads, "q"] \arrow[d, "g"'] & C \arrow[d, "h"'] \\ A' \arrow[r, tail, "j'"'] & B' \arrow[r, two heads, "q'"'] & C' \end{tikzcd}$$

Then there is a unique homomorphism $\delta\colon \ker(h) \to \mathrm{cok}(f)$ *such that* $\delta(q(b)) = a' + f(A)$ *whenever* $g(b) = j'(a')$. *Moreover, this fits into an exact sequence*

$$0 \to \ker(f) \xrightarrow{j} \ker(g) \xrightarrow{q} \ker(h) \xrightarrow{\delta} \mathrm{cok}(f) \xrightarrow{j} \mathrm{cok}(g) \xrightarrow{q} \mathrm{cok}(h) \to 0.$$

Proof. We can construct a cochain complex U^* by putting $U^0 = A$ and $U^1 = A'$ and $U^i = 0$ for all $i \notin \{0, 1\}$. The differential $U^0 \to U^1$ is just f, and all other differentials are zero for trivial reasons. We construct V^* from g and W^* from h in the same way. Note that $H^0(U^*) = \ker(f)$ and $H^1(U^*) = \mathrm{cok}(f)$ and all other cohomology groups vanish, and similarly for V^* and W^*. The maps j and j' give a cochain map $U^* \to V^*$, and the maps q and q' give a cochain map $V^* \to W^*$, and by putting these together we get a short exact sequence of cochain complexes. The claim now follows by applying Proposition 1.24. $\square$

Proposition 1.30. *Suppose we have a commutative diagram of cochain complexes and cochain maps as follows, in which the rows are short exact:*

$$\begin{tikzcd} U^* \arrow[r, "\phi"] \arrow[d, "\lambda"'] & V^* \arrow[r, "\psi"] \arrow[d, "\mu"'] & W^* \arrow[d, "\nu"'] \\ P^* \arrow[r, "\rho"'] & Q^* \arrow[r, "\sigma"'] & R^*. \end{tikzcd}$$

Then the following square also commutes:

$$\begin{tikzcd} H^k(W^*) \arrow[r, "\delta"] \arrow[d, "\nu_*"'] & H^{k+1}(U^*) \arrow[d, "\lambda_*"] \\ H^k(R^*) \arrow[r, "\delta"'] & H^{k+1}(P^*). \end{tikzcd}$$

Proof. If $(\overline{w}, w, v, u, \overline{u})$ is a snake for the top row, then $(\nu_*(\overline{w}), \nu(w), \mu(v), \lambda(u), \lambda_*(\overline{u}))$ is a snake for the bottom row. □

Corollary 1.31. *In the context of Proposition 1.30, if two of the three maps*

$$\lambda_* \colon H^*(U^*) \to H^*(P^*) \qquad \mu_* \colon H^*(V^*) \to H^*(Q^*) \qquad \nu_* \colon H^*(W^*) \to H^*(R^*)$$

are isomorphisms, then so is the third.

Remark 1.32. When we say that $\lambda_* \colon H^*(U^*) \to H^*(P^*)$ is an isomorphism, we really mean that the map $\lambda_* \colon H^k(U^*) \to H^k(P^*)$ is an isomorphism for all k. From the proof below one can easily extract some slightly more refined statements, involving particular values of k.

Proof. Suppose that λ_* and μ_* are isomorphisms. More precisely, we assume that $\lambda_* \colon H^k(U^*) \to H^k(P^*)$ is an isomorphism for all k, and similarly for μ. Consider the diagram

$$\begin{tikzcd} H^k(U^*) \arrow[r, "\phi_*"] \arrow[d, "\lambda_*"', "\simeq"] & H^k(V^*) \arrow[r, "\psi_*"] \arrow[d, "\mu_*"', "\simeq"] & H^k(W^*) \arrow[r, "\delta"] \arrow[d, "\nu_*"'] & H^{k+1}(U^*) \arrow[r, "\phi_*"] \arrow[d, "\simeq"', "\lambda_*"] & H^{k+1}(V^*) \arrow[d, "\simeq"', "\mu_*"] \\ H^k(P^*) \arrow[r, "\rho_*"] & H^k(Q^*) \arrow[r, "\sigma_*"] & H^k(R^*) \arrow[r, "\delta"] & H^{k+1}(P^*) \arrow[r, "\rho_*"] & H^{k+1}(Q^*). \end{tikzcd}$$

The middle square commutes by Proposition 1.30, and the other two squares commute by Proposition 1.17. It follows by the five lemma (Lemma 1.23) that ν_* is an isomorphism, as claimed.

This completes the case where λ_* and μ_* are assumed to be isomorphisms. The other two cases can be proved in the same way, after extending the above ladder diagram two steps to the left. □

We next want to examine how connecting homomorphisms interact with ring structures on cochain complexes.

Definition 1.33. Let A^* be a differential graded ring. A *differential A^*-module* is a cochain complex U^* with a multiplication rule $A^i \times U^j \to U^{i+j}$ satisfying the obvious associativity and distributivity conditions, such that $d(au) = d(a)u + (-1)^i a\, d(u)$ for all $a \in A^i$ and $u \in U^j$. By an evident analogue of Proposition 1.9, this makes $H^*(U^*)$ into a module over $H^*(A^*)$.

Proposition 1.34. *Let A^* be a differential graded ring, and let $U^* \xrightarrow{\phi} V^* \xrightarrow{\psi} W^*$ be a short exact sequence of differential graded A^*-modules (so $\phi(au) = a\phi(u)$ for all $a \in A^j$ and $u \in U^k$, and similarly for ψ). Then for $\overline{a} \in H^j(A^*)$ and $\overline{u} \in H^k(W^*)$ we have $\delta(\overline{au}) = (-1)^j \overline{a}\delta(\overline{u})$.*

Proof. Choose a cocycle $a \in A^j$ representing $\overline{a}$ (so $da = 0$) and a snake $(\overline{w}, w, v, u, \overline{u})$ showing that $\delta(\overline{w}) = \overline{u}$. One of the snake conditions is that $dv = \phi(u)$, which implies that $d(av) = d(a)v + (-1)^j a\, d(v) = (-1)^j a\, d(v) = \phi((-1)^j au)$. It follows that $(\overline{aw}, aw, av, (-1)^j au, (-1)^j \overline{au})$ is a snake, which implies the claim. □

Example 1.35. Let A^* be a differential graded ring, and let I^* and J^* be ideals in A^* such that $d(I^*) \leq I^*$ and $d(J^*) \leq J^*$. We can then consider A^*/I^*, A^*/J^*, $A^*/(I^* + J^*)$ and $A^*/(I^* \cap J^*)$ as differential graded rings. We can also define maps

$$A^*/(I^* \cap J^*) \xrightarrow{\phi} (A^*/I^*) \times (A^*/J^*) \xrightarrow{\psi} A^*/(I^* + J^*)$$

by

$$\phi(a + (I^* \cap J^*)) = (a + I^*, a + J^*)$$
$$\psi(b + I^*, c + J^*) = b - c + (I^* + J^*).$$

It is straightforward to check that these are well-defined, and that the displayed sequence is short exact. The map ϕ is a homomorphism of differential graded rings. The map ψ is not a ring homomorphism, but both ϕ and ψ can be regarded as homomorphisms of differential graded modules over A^*. We thus obtain an exact sequence

$$H^k(A^*/(I^* \cap J^*)) \xrightarrow{\phi_*} H^k(A^*/I^*) \times H^k(A^*/J^*) \xrightarrow{\psi_*} H^k(A^*/(I^* + J^*)) \xrightarrow{\delta} H^{k+1}(A^*/(I^* \cap J^*)) \xrightarrow{\phi_*} H^{k+1}(A^*/I^*) \times H^{k+1}(A^*/J^*),$$

in which $\delta(\overline{a}\overline{b}) = (-1)^j \overline{a}\delta(\overline{b})$ for all $\overline{a} \in H^j(A^*)$ and $\overline{b} \in H^k(A^*/(I^* + J^*))$. We call this the *algebraic Mayer-Vietoris sequence*. The traditional Mayer-Vietoris sequence for the cohomology of topological spaces can be constructed as a special case.

Remark 1.36. Here $H^*(I^*)$ cannot generally be regarded as a subgroup of $H^*(A^*)$. Instead, we have a short exact sequence $I^* \xrightarrow{j} A^* \xrightarrow{q} A^*/I^*$ of differential A^*-modules, giving rise to another long exact sequence

$$H^{k-1}(A^*) \xrightarrow{q_*} H^{k-1}(A^*/I^*) \xrightarrow{\delta} H^k(I^*) \xrightarrow{i_*} H^k(A^*) \xrightarrow{q_*} H^k(A^*/I^*) \xrightarrow{\delta} H^{k+1}(I^*).$$

Using exactness, we see that i_* is injective iff $\delta = 0$ iff q_* is surjective. If so, then i_* identifies $H^*(I^*)$ with an ideal in $H^*(A^*)$ and q_* induces an isomorphism $H^*(A^*)/H^*(I^*) \simeq H^*(A^*/I^*)$. In practice one often encounters examples like this where $\delta = 0$, but one also often encounters examples where $\delta \neq 0$.

2. Chain complexes

As well as cochain complexes, we will sometimes need the very similar theory of chain complexes, which we now describe.

Definition 2.1. A *chain complex* is a system of abelian groups C_k (for $k \in \mathbb{Z}$) equipped with homomorphisms $d_k^C : C_k \to C_{k-1}$ (called *differentials*) such that the composites

$$C_{k+1} \xrightarrow{d_{k+1}^C} C_k \xrightarrow{d_k^C} C_{k-1}$$

are all zero. We say that C_* is *nonnegative* if $C_k = 0$ for all $k < 0$. (This will be the usual case for us.)

There are two different ways we can relate chain complexes to cochain complexes. The first is as follows:

Definition 2.2. Given a chain complex C_*, we define a graded abelian group $(TC)^*$ by $(TC)^k = C_{-k}$. For each element $c \in C_{-k}$ we have a corresponding element in $(TC)^k$; it will be notationally convenient to denote that element by Tc rather than c. We can then introduce a differential $d\colon (TC)^k \to (TC)^{k+1}$ by $d(Tc) = T(dc)$, which makes $(TC)^*$ into a cochain complex. We call it the *twist* of C_*.

It is clear that this construction gives a perfect equivalence between chain complexes and cochain complexes, so all definitions and results from Section 1 have analogues for chain complexes. We will briefly discuss these analogues below, in order to pin down the various grading conventions.

First, however, we describe the other main way to relate chains and cochains.

Definition 2.3. Given a chain complex A_*, we define a graded abelian group $(DA)^*$ by $(DA)^k = \mathrm{Hom}(A_k, \mathbb{Z})$. Given a homomorphism $u\colon A_k \to \mathbb{Z}$, we define $du\colon A_{k+1} \to \mathbb{Z}$ by $(du)(x) = (-1)^k u(dx)$. This gives a homomorphism $d\colon (DA)^k \to (DA)^{k+1}$. This makes $(DA)^*$ into a cochain complex, which we call the *dual* of A.

Remark 2.4. Let A be a free abelian group. If A is finitely generated, then $\mathrm{Hom}(A, \mathbb{Z})$ is also a free abelian group of the same finite rank, and the natural map $\kappa\colon A \to \mathrm{Hom}(\mathrm{Hom}(A, \mathbb{Z}), \mathbb{Z})$ (given by $\kappa(a)(u) = u(a)$) is an isomorphism. Thus, we can pass freely between A and $\mathrm{Hom}(A, \mathbb{Z})$ with no loss of information. However, we will typically work with chain complexes A_* for which each term A_k is a free abelian groups of infinite rank, and thus is the direct sum of an infinite number of copies of $\mathbb{Z}$. This means that the dual is an infinite *product* of copies of $\mathbb{Z}$, and so is not free abelian. It also turns out that the map $\kappa\colon A_k \to \mathrm{Hom}(\mathrm{Hom}(A_k, \mathbb{Z}), \mathbb{Z})$ is injective but not surjective. Thus, we will need to work harder in this case to understand the relationship between A_* and $(DA)^*$.

Definition 2.5. Let C_* be a chain complex. We put

$$\begin{aligned} Z_k(C_*) &= \ker(d^C_k : C_k \to C_{k-1}) \\ B_k(C_*) &= \mathrm{image}(d^C_{k+1} \colon C_{k+1} \to C_k) \\ H_k(C_*) &= Z_k(C_*)/B_k(C_*). \end{aligned}$$

(For the definition of $H_k(C_*)$ to make sense, we need to know that $B_k(C_*) \leq Z_k(C_*)$. This follows immediately from the condition $d^2 = 0$.) The elements of $Z_k(C_*)$ are called *cycles*, and the elements of $B_k(C_*)$ are called *boundaries*. The groups $H_k(C_*)$ are the *homology groups* of the complex.

Definition 2.6. Let A_* and B_* be chain complexes. A *chain map* from A_* to B_* is a sequence of maps $\phi_k \colon A_k \to B_k$ for all $k \in \mathbb{Z}$ such that the following diagrams commute:

$$\begin{tikzcd} A_k \arrow[r, "\phi_k"] \arrow[d, "d_A"'] & B_k \arrow[d, "d_B"] \\ A_{k-1} \arrow[r, "\phi_{k-1}"'] & B_{k+1} \end{tikzcd}$$

Where there is no danger of confusion, we will suppress all subscripts and superscripts, so the commutation condition is just $d\phi = \phi d$.

By an evident analogue of Proposition 1.17 we see that a chain map $\phi \colon A_* \to B_*$ induces a homomorphism $\phi_* \colon H_*(A) \to H_*(B)$ in a functorial way.

Definition 2.7. Let $\phi, \psi \colon A_* \to B_*$ be chain maps between chain complexes. A *chain homotopy* from ϕ to ψ is a system of maps $\sigma_k \colon A_k \to A_{k+1}$ such that $\psi_k - \phi_k = \sigma_{k-1}d_k + d_{k+1}\sigma_k$ (or more briefly, $\psi - \phi = d\sigma + \sigma d$). We say that ϕ and ψ are *chain homotopic* (written $\phi \simeq \psi$) if there exists a cochain homotopy between them.

As in the case of cochain maps, we see that chain homotopy is an equivalence relation that is compatible with composition, and that chain homotopic maps induce the same map of homology groups.

Definition 2.8. Given a chain map $\phi \colon A_* \to B_*$ as above, we define $D(\phi) \colon D(B)^k \to D(A)^k$ by $D(\phi)(v) = v \circ \phi_k$. This is easily seen to be a cochain map. We will write ϕ^* for the map $H^k(D(B)) \to H^k(D(A))$ induced by ϕ. If we have another chain map $\psi \colon A_* \to B_*$ and a chain homotopy σ from ϕ to ψ, we define $(D\sigma)^k \colon (DB)^k \to (DA)^{k-1}$ by $(D\sigma)(v) = (-1)^{k-1} v \circ \sigma_{k-1}$. One can check directly that this gives a cochain homotopy from $D(\phi)$ to $D(\psi)$.

Remark 2.9. Given chain maps $A_* \xrightarrow{\phi} B_* \xrightarrow{\psi} C_*$, it is easy to see that $D(\psi\phi) = D(\phi) \circ D(\psi)$ and $(\psi\phi)^* = \phi^*\psi^*$ on cohomology. In other words, $D(A)^*$ and $H^*(D(A))$ are contravariant functors of A_*.

Proposition 2.10. *Let A_* be a chain complex. Then there is a well-defined bilinear pairing $H^k(DA) \times H_k(A) \to \mathbb{Z}$ given by*

$$\langle u + B^k(DA), a + B_k(A) \rangle = u(a)$$

(for $u \in Z^k(DA)$ and $a \in Z_k(A)$). Moreover, for any chain map $\phi \colon A_* \to B_*$ and any $\alpha \in H_k(A)$ and any $\beta \in H^k(DB)$ we have

$$\langle \phi_*(\alpha), \beta \rangle = \langle \alpha, \phi^*(\beta) \rangle.$$

Proof. For the pairing to be well-defined, we need to check that $(u + dx)(a + dt) = u(a)$ for all $x \in (DA)^{k-1}$ and $t \in A_{k+1}$. By definition of the differential on DA, we have $dx(a + dt) = (-1)^{k-1}x(d(a + dt))$ but $da = 0$ and $d^2t = 0$ so $dx(a + dt) = 0$. We also have $u(dt) = (-1)^k(du)(t)$ but $du = 0$ by assumption so $u(dt) = 0$. From this it is clear that $(u + dx)(a + dt) = u(a)$ as required. The statement about (co)chain maps follows directly from the definitions. $\square$

The next two propositions can be deduced from the corresponding results in Section 1 using the twisting functor from Definition 2.2, or they can be proved by a slight adaptation of the proofs for those earlier results.

Proposition 2.11. *Suppose we have a short exact sequence* $U_* \xrightarrow{\phi} V_* \xrightarrow{\psi} W_*$ *of chain complexes. Then there is a natural system of maps* $\delta_k \colon H_k(W^*) \to H_{k-1}(U^*)$ *(called* connecting homomorphisms *such that the sequence*

$$H_k(U_*) \xrightarrow{\phi_*} H_k(V_*) \xrightarrow{\psi_*} H_k(W_*) \xrightarrow{\delta} H_{k-1}(U_*) \xrightarrow{\phi_*} H_{k-1}(V_*)$$

is exact for all k. □

Proposition 2.12. *Suppose we have a commutative diagram of chain complexes and chain maps as follows, in which the rows are short exact:*

$$\begin{tikzcd}
U_* \arrow[r, "\phi"] \arrow[d, "\lambda"'] & V_* \arrow[r, "\psi"] \arrow[d, "\mu"'] & W_* \arrow[d, "\nu"'] \\
P_* \arrow[r, "\rho"'] & Q_* \arrow[r, "\sigma"'] & R_*.
\end{tikzcd}$$

Then the following square also commutes:

$$\begin{tikzcd}
H_k(W_*) \arrow[r, "\delta"] \arrow[d, "\nu_*"'] & H_{k-1}(U_*) \arrow[d, "\lambda_*"] \\
H_k(R_*) \arrow[r, "\delta"'] & H_{k-1}(P_*).
\end{tikzcd}$$

Thus, if two of the three maps

$$\lambda_* \colon H_*(U_*) \to H_*(P_*) \qquad \mu_* \colon H_*(V_*) \to H_*(Q_*) \qquad \nu_* \colon H_*(W_*) \to H_*(R_*)$$

are isomorphisms, then so is the third. □

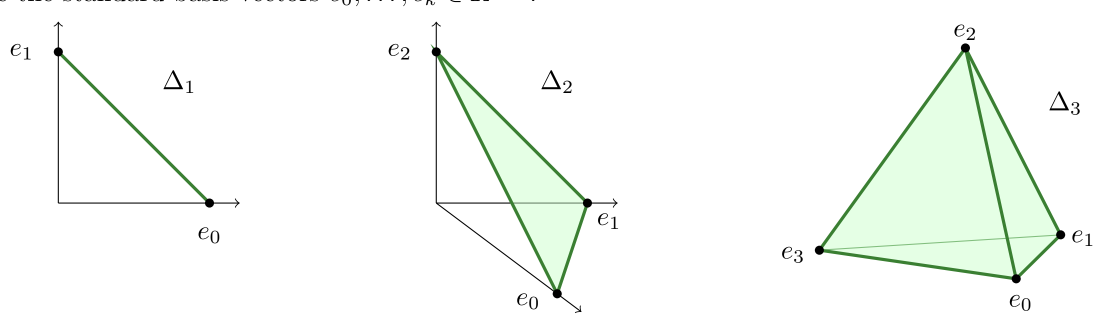
3. Singular homology and cohomology

We now give the definition of the cohomology groups $H^*(X)$. It will be technically convenient to treat the homology groups $H_*(X)$ at the same time, even though we have not emphasised these in the main text. These are less richly structured than the cohomology groups because they do not have a ring structure, but nonetheless they are often useful. For almost all the spaces that we have discussed, we just have $H_k(X) = \mathrm{Hom}(H^k(X), \mathbb{Z})$ and $H^k(X) = \mathrm{Hom}(H_k(X), \mathbb{Z})$, so homology and cohomology determine each other in a straightforward way. The relationship can be a little more complicated for general X, however.

Definition 3.1. First, we put

$$\Delta_k = \{(x_0, \dots, x_k) \in [0,1]^d \mid \sum_i x_i = 1\}.$$

We find that Δ_0 is a point, Δ_1 is a line segment, Δ_2 is a triangle and Δ_3 is a tetrahedron. The vertices of Δ_k are the standard basis vectors $e_0, \dots, e_k \in \mathbb{R}^{k+1}$.

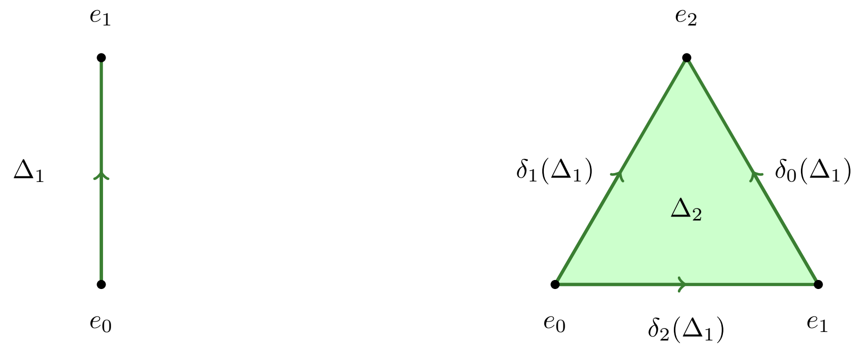


In general, Δ_k is called a *k-simplex*.

Definition 3.2. Now suppose that $0 \leq i \leq d$. We can define a map $\delta_i \colon \Delta_{d-1} \to \Delta_d$ by

$$\delta_i(x) = (x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{d-1}).$$

For example, we have three maps δ_0, δ_1 and δ_2 from the line segment Δ_1 to the triangle Δ_2; each of these identifies the line segment with one of the edges of the triangle.



Remark 3.3. Suppose that X is a subspace of a vector space. We say that a map $u\colon \Delta_n \to X$ is *affine* if for all $x \in \Delta_n$ we have $u(x) = u(\sum_i x_i e_i) = \sum_i x_i\, u(e_i)$ (where as usual e_i denotes the i'th standard basis vector). We can describe $\delta_i\colon \Delta_{d-1} \to \Delta_d$ as the unique affine map such that

$$\delta_i(e_j) = \begin{cases} e_j & \text{if } 0 \leq j < i \\ e_{j+1} & \text{if } i \leq j < d. \end{cases}$$

We also see from this that $\delta_i(\Delta_{d-1})$ is the face of Δ_d opposite the vertex e_i.

Definition 3.4. Now let $S_k(X)$ denote the set of all continuous maps $u\colon \Delta_k \to X$. Thus $S_0(X)$ is just the set X itself, whereas $S_1(X)$ is the set of continuous paths in X. For bookkeeping convenience, we take $S_k(X)$ to be the empty set for $k < 0$.

Definition 3.5. Next, we write $C_k(X)$ for the free abelian group generated by $S_k(X)$. Thus, for each $u \in S_k(X)$ we have a basis element $[u] \in C_k(X)$, and every element $m \in C_k(X)$ can be written as a finite linear combination $m = \sum_{i=1}^d m_i[u_i]$ for some integers $m_i \in \mathbb{Z}$ and some maps $u_i\colon \Delta_k \to X$. The elements of $C_k(X)$ are called *singular k-chains* on X. We define a homomorphism $d\colon C_k(X) \to C_{k-1}(X)$ (called the *boundary homomorphism*) by

$$d([u]) = \sum_{i=0}^{k} (-1)^i [u \circ \delta_i]$$

(or $d = 0$ when $k \leq 0$).

Remark 3.6. Note that we have only specified the effect of d on the basis elements, but this is sufficient: for a general element $m = \sum_j m_j[u_j] \in C_k(X)$, we can and must put

$$d(m) = \sum_j m_j d([u_j]) = \sum_j \sum_{i=0}^{k} (-1)^i m_j [u_j \circ \delta_i].$$

Remark 3.7. Recall that $S_0(X) = X$, so $C_0(X)$ is the free abelian group on the points of X. Now consider a map $u\colon \Delta_1 \to X$. The simplex Δ_1 is just an interval joining the point $e_0 = (1, 0)$ to the point $e_1 = (0, 1)$, so u can be thought of as a path joining $u(e_0)$ to $u(e_1)$ in X. We have $d([u]) = [u(e_1)] - [u(e_0)]$, which is the formal difference between the endpoints of the path.

We next claim that $d(d(m))$ is always zero. This relies on the following lemma.

Lemma 3.8. *If $0 \leq j \leq i \leq k$ then $\delta_j\delta_i = \delta_{i+1}\delta_j\colon \Delta_{k-1} \to \Delta_{k+1}$.*

Proof. Consider a point $x = (x_0, \ldots, x_{k-1}) \in \Delta_{k-1}$. Then $\delta_i(x)$ is the same as x but with a zero inserted in position i. If we now apply δ_j, this inserts another zero in position j. As $j \leq i$ this moves the first zero into position $i+1$. Suppose instead that we first do δ_j, inserting a zero in position j. If we then apply δ_{i+1} then we get another zero in position $i+1$, and this insertion does not move the first zero because $i+1 > j$. The claim follows.

For example, in the case where $k = 6$, $j = 2$ and $i = 4$ we have

$$\begin{aligned}
\delta_2\delta_4(x_0, x_1, x_2, x_3, x_4, x_5) &= \delta_2(x_0, x_1, x_2, x_3, 0, x_4, x_5) \\
&= (x_0, x_1, 0, x_2, x_3, 0, x_4, x_5) \\
\delta_5\delta_2(x_0, x_1, x_2, x_3, x_4, x_5) &= \delta_5(x_0, x_1, 0, x_2, x_3, x_4, x_5) \\
&= (x_0, x_1, 0, x_2, x_3, 0, x_4, x_5).
\end{aligned}$$

□

Corollary 3.9. *The composite $C_{k+1}(X) \xrightarrow{d} C_k(X) \xrightarrow{d} C_{k-1}(X)$ is zero, so $C_*(X)$ is a chain complex.*

Proof. It will suffice to show that $d(d([v])) = 0$ for all $v \in S_{k+1}(X)$. Put $m_{ji} = (-1)^{i+j}[v \circ \delta_j \circ \delta_i]$; we must show that $\sum_{i=0^k} \sum_{j=0}^{k+1} m_{ji} = 0$. The lemma tells us that when $0 \leq j \leq i \leq k$ we have $m_{j,i} + m_{i+1,j} = 0$. Consider the case $k = 3$. The relevant terms are then as follows:

$$\begin{array}{ccccc}
{\color{red}m_{00}} & {\color{blue}m_{10}} & {\color{blue}m_{20}} & {\color{blue}m_{30}} & {\color{blue}m_{40}} \\
{\color{red}m_{01}} & {\color{red}m_{11}} & {\color{blue}m_{21}} & {\color{blue}m_{31}} & {\color{blue}m_{41}} \\
{\color{red}m_{02}} & {\color{red}m_{12}} & {\color{red}m_{22}} & {\color{blue}m_{32}} & {\color{blue}m_{42}} \\
{\color{red}m_{03}} & {\color{red}m_{13}} & {\color{red}m_{23}} & {\color{red}m_{33}} & {\color{blue}m_{43}}
\end{array}$$

We find that m_{00} cancels m_{10}, m_{01} cancels m_{20}, and in general each column below the dividing line cancels with the row of the same length above the dividing line. The same pattern works for all k; we leave the formal proof to the reader. □

Definition 3.10. We define the homology groups of the space X to be the homology groups of the chain complex $C_*(X)$, so

$$H_k(X) = H_k(C_*(X)) = \frac{\ker(d\colon C_k(X) \to C_{k-1}(X))}{\mathrm{image}(d\colon C_{k+1}(X) \to C_k(X)}.$$

We also write $C^*(X)$ for the dual cochain complex $D(C_*(X))$ as in Definition 2.3. We then put

$$H^k(X) = H^k(C^*(X)) = \frac{\ker(d\colon C^k(X) \to C^{k+1}(X))}{\mathrm{image}(d\colon C^{k-1}(X) \to C^k(X)}.$$

Remark 3.11. As $C_k(X)$ is freely generated by $S_k(X)$, any homomorphism from $C_k(X)$ to $\mathbb{Z}$ is determined by its values on the basis elements in $S_k(X)$, and these values can be precribed arbitrarily. We can thus identify $C^k(X) = \mathrm{Hom}(C_k(X), \mathbb{Z})$ with the set $\mathrm{Map}(S_k(X), \mathbb{Z})$ of all functions from $S_k(X)$ to $\mathbb{Z}$. In this picture, the differential $d\colon C^k(X) \to C^{k+1}(X)$ can be described as follows. Suppose that $u \in C^k(X)$, so we have $u(s) \in \mathbb{Z}$ for all $s\colon \Delta_k \to X$. We must define $du \in C^{k+1}(X)$, or equivalently we must define $(du)(t)$ for all $t\colon \Delta_{k+1} \to X$. We can compose t with the maps $\delta_i\colon \Delta_k \to \Delta_{k+1}$ to get maps $v \circ \delta_i \in S_k(X)$ and thus integers $u(v \circ \delta_i) \in \mathbb{Z}$. After chasing through the various identifications we find that the appropriate definition is

$$(du)(t) = \sum_{i=0}^{k+1} (-1)^{k+i} u(v \circ \delta_i).$$

Remark 3.12. Consider again the definition $C^k(X) = \mathrm{Map}(S_k(X), \mathbb{Z})$. As always, we are assuming that X has a topology, and $S_k(X)$ is the set of continuous maps from Δ_k to X. It would be possible to introduce a sensible topology on $S_k(X)$, but we have not done so, and it would not be useful for our present purposes. Instead, we just consider $S_k(X)$ as a discrete set, and $C^k(X)$ is the set of *all* maps from $S_k(X)$ to $\mathbb{Z}$, with no continuity condidtion. This does mean that $C^k(X)$ is extravagantly large, and the subgroups $Z^k(X)$ and $B^k(X)$ (of cocycles and coboundaries) are also extravagantly large. Nonetheless, the quotient $H^k(X) = Z^k(X)/B^k(X)$ is often manageably small.

The most basic calculation is as follows:

Proposition 3.13. *Suppose that X is a set with the discrete topology. Then*

(a) *$H_0(X)$ is the free abelian group generated by X.*
(b) *$H^0(X) = \mathrm{Map}(X, \mathbb{Z})$.*
(c) *For $k \neq 0$ we have $H_k(X) = 0$ and $H^k(X) = 0$.*

In particular, if X is a single point then $H_0(X) = H^0(X) = \mathbb{Z}$.

Proof. In the light of Remarks 3.7 and 3.11, we can rephrase (a) and (b) as saying that $H_0(X) = C_0(X)$ and $H^0(X) = C^0(X)$. As X is discrete, every continuous map $\Delta_k \to X$ is constant. Thus, for $k \geq 0$ we can identify $S_k(X)$ with X, and $C_k(X)$ with $C_0(X)$, and $C^k(X)$ with $C^0(X)$. With this identification, the boundary map $d\colon C_{k+1}(X) \to C_k(X)$ is the identity on $C_0(X)$ multiplied by $\sum_{i=0}^{k+1}(-1)^i$, which is zero when k is even, and one when k is odd. In other words, the complex

$$C_0(X) \xleftarrow{d} C_1(X) \xleftarrow{d} C_2(X) \xleftarrow{d} C_3(X) \xleftarrow{d} C_4(X) \xleftarrow{d} C_5(X) \xleftarrow{d} \dots$$

is just

$$C_0(X) \xleftarrow{0} C_0(X) \xleftarrow{1} C_0(X) \xleftarrow{0} C_0(X) \xleftarrow{1} C_0(X) \xleftarrow{0} C_0(X) \xleftarrow{1} \dots .$$

This means that when k is odd we have $Z_k(X) = B_k(X) = 0$, whereas if k is even and $k > 0$ we have $Z_k(X) = B_k(X) = 0$; either way, we have $H_k(X) = 0$. Finally, we have $Z_0(X) = C_0(X)$ and $B_0(X) = 0$ so $H_0(X) = C_0(X)$ as claimed. The claims about $H^*(X)$ follow in the same way, after dualising the above description of $C_*(X)$. □

For spaces that are not discrete, we can still give an elementary account of $H_0(X)$ and $H^0(X)$.

Definition 3.14. We write $\pi_0(X)$ for the set of path-components of X. In more detail, we introduce a relation on X by $x_0 \sim x_1$ if there exists a continuous path $u\colon [0,1] \to X$ with $u(0) = x_0$ and $u(1) = x_1$. This is easily seen to be an equivalence relation, and we write $\pi_0(X)$ for the set of equivalence classes.

Proposition 3.15. *Let X be an arbitrary space. Then*

(a) *$H_0(X)$ can be identified with $\mathbb{Z}[\pi_0(X)]$, the free abelian group generated by the set $\pi_0(X)$.*
(b) *$H^0(X) = \mathrm{Map}(\pi_0(X), \mathbb{Z})$.*

In particular, if X is path-connected then $H_0(X) = H^0(X) = \mathbb{Z}$.

Proof. We will start with the cohomological statement. We have $S_0(X) = X$ so $C^0(X)$ is the set of all (possibly discontinuous) maps $u\colon X \to \mathbb{Z}$. We have $C^{-1}(X) = 0$ so $B^0(X) = 0$ so $H^0(X) = Z^0(X) = \ker(d\colon C^0(X) \to C^1(X))$. For any $s\colon \Delta_1 \to X$, the map $s \circ \delta_0\colon \Delta_0 \to X$ corresponds to the point $s(e_1)$, and similarly $s \circ \delta_1$ corresponds to $s(e_0)$, so $(du)(s) = s(e_1) - s(e_0)$. Note here that s can be regarded as a path in X from $s(e_0)$ to $s(e_1)$. Thus, we see that $du = 0$ iff u takes the same value on the endpoints of every path, and so factors through the quotient set $\pi_0(X)$. This proves that $H^0(X) = \mathrm{Map}(\pi_0(X), \mathbb{Z})$ as claimed.

The homological statement is similar but needs a little more bookkeeping. First, observe that $C_{-1}(X) = 0$ so $Z_0(X) = C_0(X)$ and $H_0(X) = C_0(X)/B_0(X) = \mathrm{cok}(d\colon C_1(X) \to C_0(X))$. Let $q\colon X \to \pi_0(X)$ be the evident map that sends a point $x \in X$ to the corresponding path component. This induces a map $C_0(X) = \mathbb{Z}[X] \to \mathbb{Z}[\pi_0(X)]$, which we also call q. This is clearly surjective. Recall that an element $u \in S_1(X)$ can be regarded as a path in X, with endpoints x and y say, and that $d([u]) = [y] - [x]$. It follows that $qd([u]) = [q(y)] - [q(x)]$, but x and y are in the same path component, so $q(y) = q(x)$, so $qd([u]) = 0$. This proves that $qd = 0$, so $q(B_0(X)) = 0$, so there is an induced map $\overline{q}\colon H_0(X) \to \mathbb{Z}[\pi_0(X)]$ given by $\overline{q}([x] + B_0(X)) = [q(x)]$. It should be reasonably clear that this is an isomorphism. A formal proof can be given as follows. For each path component $c \in \pi_0(X)$ we can choose a point $f(c) \in X$ representing that component. This gives a map $f\colon \pi_0(X) \to X$ with $qf = 1_{\pi_0(X)}$. We again extend this to give a homomorphism $\mathbb{Z}[\pi_0(X)] \to \mathbb{Z}[X] = C_0(X)$ by putting $f(\sum_i n_i[c_i]) = \sum_i n_i[f(c_i)]$, and we still have $qf = 1$. Next, for each $x \in X$ we note that $fq(x)$ is in the same path component as x, so we can choose a path $g(x) \in S_1(X)$ with $g(x)(e_0) = fq(x)$ and $g(x)(e_1) = x$. This gives a map $g\colon S_0(X) \to S_1(X)$, and thus a homomorphism $C_0(X) \to C_1(X)$. By construction, we have $dg([x]) = [x] - fq([x])$. As the elements $[x]$ form a basis, this means that $dg = 1 - fq$ as homomorphisms $C_0(X) \to C_1(X)$. In particular, if $u \in \ker(q)$ then $dg(u) = u - fq(u) = u$, so $u \in B_1(X)$. It follows that the induced map $\overline{q}$ on $C_0(X)/B_1(X)$ is injective, and thus is an isomorphism as required. □

Remark 3.16. More generally, let $(X_i)_{i\in I}$ be the family of all path components of X. We then find that any map $\Delta_k \to X$ factors through one of the sets X_i, so $S_k(X) = \coprod_i S_k(X_i)$ and $C_k(X) = \bigoplus_i C_k(X_i)$ and $C^k(X) = \prod_i C^k(X_i)$. The differentials respect these splittings and we deduce that $H_k(X) = \bigoplus_i H_k(X_i)$ and $H^k(X) = \prod_i H^k(X_i)$.

The following result is a special case of the homotopy invariance of homology, which will be proved later. However, the special case has a much simpler proof, so it is illuminating to discuss it separately.

Proposition 3.17. *Suppose we have a space X with a basepoint 0_X, and a contraction $h\colon [0,1]\times X \to X$ with $h(0,x) = 0_X$ and $h(1,x) = x$ for all $x \in X$. Then $H_0(X) = \mathbb{Z}$, and $H_k(X) = 0$ for all $k \neq 0$.*

Proof. First, we can define $m\colon [0,1]\times\Delta_k \to \Delta_{k+1}$ by $m(r,t) = (1-r, rt)$. This is surjective, and it collapses $\{0\}\times\Delta_k$ to the point $e_0 = (1,0) \in \Delta_{k+1}$, but is otherwise injective. It therefore induces a continuous bijection from $([0,1]\times\Delta_k)/(\{0\}\times\Delta_k)$ to Δ_{k+1}, which is a homeomorphism because the source is compact and the target is Hausdorff. Geometrically, this just means that Δ_{k+1} can be regarded as a cone on Δ_k.

Next, we define a map $s\colon S_k(X) \to S_{k+1}(X)$ as follows. A point in $S_k(X)$ is a map $u\colon \Delta_k \to X$. The map $s(u)\colon \Delta_{k+1} \to X$ is given by $s(u)(e_0) = 0_X$, and

$$s(u)(t) = h(1-t_0, u(t_1/(1-t_0)\dots, t_{k+1}/(1-t_0)))$$

for $t \neq e_0$. In other words, $s(u)$ is the unique map such that the following diagram commutes:

$$\begin{array}{ccc} [0,1]\times\Delta_k & \xrightarrow{m} & \Delta_{k+1} \\ {\scriptstyle 1\times u}\downarrow & & \downarrow{\scriptstyle s(u)} \\ [0,1]\times X & \xrightarrow[h]{} & X. \end{array}$$

This means that $s(u)\circ m$ is continuous, and m is a quotient map, so $s(u)$ is continuous, and so can be regarded as an element of $S_{k+1}(X)$. We have thus defined a map $s\colon S_k(X) \to S_{k+1}(X)$, which induces a map $s\colon C_k(X) \to C_{k+1}(X)$ of the corresponding free abelian groups.

We next analyse the maps $s(u)\circ\delta_i\colon \Delta_k \to X$ for $0 \leq i \leq k+1$. When $i = 0$ we have

$$s(u)(\delta_0(t)) = s(u)(0,t) = h(1, u(t)) = u(t),$$

so $s(u)\circ\delta_0 = u$. When $i = 1$ we have

$$\begin{aligned} s(u)(\delta_1(t)) &= s(u)(t_0, 0, t_1, \dots, t_k) \\ &= h(1-t_0, u(0, t_1/(1-t_0), \dots, t_k/(1-t_0))) \\ &= h(1-t_0, (u\circ\delta_0)(t_1/(1-t_0), \dots, t_k/(1-t_0))) \\ &= s(u\circ\delta_0)(t). \end{aligned}$$

We conclude that $s(u)\circ\delta_1 = s(u\circ\delta_0)$. For $1 < i \leq k+1$ the notation would be more cumbersome if we wrote out the details, but the idea is the same, and the result is that $s(u)\circ\delta_i = s(u\circ\delta_{i-1})$. It follows that for $u \in S_k(X)$ with $k > 0$, we have

$$\begin{aligned} d(s([u])) + s(d([u])) &= \sum_{i=0}^{k+1}(-1)^i[s(u)\circ\delta_i] + \sum_{i=0}^{k}(-1)^i[s(u\circ\delta_i)] \\ &= [u] + \sum_{i=1}^{k+1}(-1)^i[s(u\circ\delta_{i-1})] + \sum_{i=0}^{k}(-1)^i[s(u\circ\delta_i)] \\ &= [u]. \end{aligned}$$

As the elements $[u]$ span $C_k(X)$, we deduce that $d(s(c)) + s(d(c)) = c$ for all $c \in C_k(X)$. In particular, for $c \in Z_k(X)$ we have $d(c) = 0$ so the relation reduces to $d(s(c)) = c$, which shows that $c \in \operatorname{image}(d) = B_k(X)$. This means that $Z_k(X) = B_k(X)$, so $H_k(X) = Z_k(X)/B_k(X) = 0$ as claimed. The argument must be modified slightly for $k = 0$, as follows. We define $\epsilon\colon C_0(X) \to \mathbb{Z}$ by $\epsilon(\sum_i n_i[x_i]) = \sum_i n_i$, and observe that $\epsilon d = 0\colon C_1(X) \to \mathbb{Z}$. For $x \in S_0(X) = X$ we find that $d(s([x])) = [x] - [0_X]$, so for $c \in C_0(X)$ we have $d(s(c)) = c - \epsilon(c)[0_X]$. From this it follows that $Z_0(X) = C_0(X) = B_0(X) \oplus \mathbb{Z}.[0_X]$, so $H_0(X) = Z_0(X)/B_0(X) = \mathbb{Z}$ as claimed. $\square$

Remark 3.18. We can rephrase the above proof as follows. Let T_* be the chain complex with $T_0 = \mathbb{Z}$ and $T_k = 0$ for all $k \neq 0$ (so the differentials are automatically zero). We can define chain maps

$$T_* \xrightarrow{\eta} C_*(X) \xrightarrow{\epsilon} T_*$$

as follows. For $k \neq 0$ we can and must take $\eta_k = 0$ and $\epsilon_k = 0$. We define $\eta_0(n) = n.[0_X] \in C_0(X)$, and $\epsilon_0(\sum_i n_i[x_i]) = \sum_i n_i$. It is straightforward to check that the composite $C_1(X) \xrightarrow{d} C_0(X) \xrightarrow{\epsilon_0} \mathbb{Z}$ is zero, and in all other cases we have $d\eta = \eta d$ and $d\epsilon = \epsilon d$ for vacuous reasons, so η and ϵ are indeed chain maps. It is also clear that $\epsilon\eta = 1_{T_*}$. The map s in the above proof gives a chain homotopy between $\eta\epsilon$ and $1_{C_*(X)}$, so $\eta_*\epsilon_*$ is the identity on $H_*(X)$. It follows that η_* and ϵ_* give mutually inverse isomorphisms between $H_*(X)$ and $H_*(T_*)$, so $H_0(X) = \mathbb{Z}$ and the other groups vanish as claimed. We remarked in Definition 2.8 that chain homotopies give rise to cochain homotopies by duality, and using this we see that $H^*(X)$ is the same as $H^*(D(T)^*)$, which again has a single $\mathbb{Z}$ in degree zero.

Example 3.19. Later we will give a general result relating $H_1(X)$ to the fundamental group $\pi_1(X)$, for an arbitrary based space X. From that result we will be able to see that $H_1(S^1) = \mathbb{Z}$. Here we will use more direct methods to construct homomorphisms $\mathbb{Z} \to H_1(S^1) \to \mathbb{Z}$ whose composite is the identity. This proves that $H_1(S^1) = \mathbb{Z} \oplus A$ for some A; in fact $A = 0$, but we will leave that for later.

First, define $\eta\colon \mathbb{R} \to S^1$ by $\eta(t) = \exp(2\pi i t)$ (so η is surjective, and $\eta(s) = \eta(t)$ iff $s - t \in \mathbb{Z}$). Define $u\colon \Delta_1 \to S^1$ by $u(t, 1-t) = \eta(t)$. Then $u(e_1) = u(e_0) = 1$, so $d([u]) = 0$, so we have an element $a = [u] + B_1(S^1) \in H_1(S^1)$. Our map $\mathbb{Z} \to H_1(S^1)$ just sends n to na.

For the other map, we will construct a diagram as follows.

$$\begin{tikzcd}
C_2(S^1) \arrow[r, "d"] \arrow[d] & C_1(S^1) \arrow[r, "d"] \arrow[d, "\omega"'] & C_0(S^1) \arrow[d, "\epsilon"] \\
0 \arrow[r] & \mathbb{R} \arrow[r, "\eta"'] & S^1.
\end{tikzcd}$$

Here d and η have been defined already, and ϵ is the homomorphism from the additive group $C_0(S^1)$ to the multiplicative group S^1 given by $\epsilon(\sum_i n_i[z_i]) = \prod_i z_i^{n_i}$. To define ω, consider an arbitrary map $v\colon \Delta_1 \to S^1$. The theory of covering spaces tells us that there exists a lifting $\widetilde{v}\colon \Delta_1 \to \mathbb{R}$ with $\eta \circ \widetilde{v} = v$, and that any two such lifts differ by an integer constant. We define the *winding number* of v to be $\omega(v) = \widetilde{v}(e_1) - \widetilde{v}(e_0) \in \mathbb{R}$. More generally, for a 1-chain $c = \sum_i m_i[v_i] \in C_1(S^1)$, we put $\omega(c) = \sum_i m_i\omega(v_i) \in \mathbb{R}$; this defines a homomorphism $\omega\colon C_1(S^1) \to \mathbb{R}$ as required. We claim that $\eta\omega = \epsilon d\colon C_1(X) \to S^1$ (so that the right hand square in the diagram commutes). Indeed, we have

$$\eta\omega([v]) = \eta(\widetilde{v}(e_1) - \widetilde{v}(e_0)) = v(e_1)/v(e_0) = \epsilon([v(e_1)] - [v(e_0)]) = \epsilon d([v])$$

as required. We also claim that $\omega d = 0\colon C_1(X) \to \mathbb{R}$. To see this, consider a map $p\colon \Delta_2 \to S^1$. As Δ_2 is homeomorphic to a square, the theory of covering spaces again gives a lift $\widetilde{p}\colon \Delta_2 \to \mathbb{R}$ with $\eta \circ \widetilde{p} = p$. The composites $\widetilde{p} \circ \delta_i\colon \Delta_1 \to \mathbb{R}$ can then be used as lifts for the paths $p \circ \delta_i\colon \Delta_1 \to S^1$, which gives

$$\begin{aligned}
\omega(p \circ \delta_0) &= \widetilde{p}(\delta_0(e_1)) - \widetilde{p}(\delta_0(e_0)) = \widetilde{p}(e_2) - \widetilde{p}(e_1) \\
\omega(p \circ \delta_1) &= \widetilde{p}(\delta_1(e_1)) - \widetilde{p}(\delta_1(e_0)) = \widetilde{p}(e_2) - \widetilde{p}(e_0) \\
\omega(p \circ \delta_2) &= \widetilde{p}(\delta_2(e_1)) - \widetilde{p}(\delta_2(e_0)) = \widetilde{p}(e_1) - \widetilde{p}(e_0) \\
\omega(d([p])) &= (\widetilde{p}(e_2) - \widetilde{p}(e_1)) - (\widetilde{p}(e_2) - \widetilde{p}(e_0)) + (\widetilde{p}(e_1) - \widetilde{p}(e_0)) = 0,
\end{aligned}$$

as claimed. This proves that the diagram above is commutative.

Now suppose that $c \in Z_1(S^1)$. We then have $\eta(\omega(c)) = \epsilon(d(c)) = \epsilon(0) = 1$, so $\omega(c) \in \mathbb{Z} \subset \mathbb{R}$. Moreover, if c actually lies in $B_1(S^1) \leq Z_1(S^1)$ then $\omega(c) = 0$, because of the relation $\omega d = 0$. It follows that there is an induced homomorphism $\overline{\omega}\colon H_1(S^1) \to \mathbb{Z}$ given by $\overline{\omega}(c + B_1(S^1)) = \omega(c)$. If $a \in H_1(S^1)$ is as defined at the beginning of this example, we see directly that $\overline{\omega}(a) = 1$, so the composite $\mathbb{Z} \to H_1(S^1) \to \mathbb{Z}$ is the identity, as promised.

Example 3.20. The boundary of Δ_n is the subspace

$$\partial\Delta_n = \{t \in \Delta_n \mid t_i = 0 \text{ for some } i\} = \bigcup_{i=0}^{n} \delta_i(\Delta_{n-1}) \subset \Delta_n.$$

The maps $\delta_i \colon \Delta_{n-1} \to \partial\Delta_n$ can be regarded as elements of $S_{n-1}(\partial\Delta_n)$, so we can define

$$c = \sum_{i=0}^{n-1} (-1)^i [\delta_i] \in C_{n-1}(\partial\Delta_n).$$

We claim that $d(c) = 0$, so $c \in Z_{n-1}(\partial\Delta_n)$. To see this, let b be the identity map $\Delta_n \to \Delta_n$, regarded as an element of $S_n(\Delta_n) \subset C_n(\Delta_n)$. We can regard $C_k(\partial\Delta_n)$ as a subgroup of $C_k(\Delta_n)$, and in that context, we have $c = d(b)$, so $d(c) = d^2(b) = 0$. We thus have an element $u = c + B_{n-1}(\partial\Delta_n) \in H_{n-1}(\partial\Delta_n)$.

Beause b is not in $C_n(\partial\Delta_n)$, the equation $d(b) = c$ does not show that $c \in B_{n-1}(\partial\Delta_n)$, so it is possible for u to be a nonzero element of $H_{n-1}(\partial\Delta_n)$. We will in fact show later that for $n > 1$, the group $H_{n-1}(\partial\Delta_n)$ is a copy of $\mathbb{Z}$, generated by u. The other homology groups are zero, apart from $H_0(\partial\Delta_n) = \mathbb{Z}$. If $n = 1$ then $\partial\Delta_n$ just consists of two points, so we again have two copies of $\mathbb{Z}$, but they both occur in dimension 0.

Note also that $\partial\Delta_n$ is homeomorphic to S^{n-1}. Indeed, we have a map $d \colon \partial\Delta_n \to \mathbb{R}^n \setminus \{0\}$ given by

$$d(t) = (t_1 - t_0, t_2 - t_1, \dots, t_n - t_{n-1}),$$

so we can define $f \colon \partial\Delta_n \to S^{n-1}$ by $f(t) = d(t)/\|d(t)\|$. It is not hard to show that this is a homeomorphism, so $H_*(S^{n-1}) \simeq H_*(\partial\Delta_n)$.

Example 3.21. As usual, we write $\Delta_n/\partial\Delta_n$ for the quotient space in which $\partial\Delta_n$ is collapsed to a single point (which we take as the basepoint). Let $u \colon \Delta_n \to \Delta_n/\partial\Delta_n$ be the obvious quotient map, and let $z \colon \Delta_n \to \Delta_n/\partial\Delta_n$ be the constant map sending everything to the basepoint. We can use these to define a chain $c = [u] - [z] \in C_n(\Delta_n/\partial\Delta_n)$. For $0 \le i \le n$ we have $u \circ \delta_i = z \circ \delta_i$, and it follows that $d(c) = 0$, so $c \in Z_n(\Delta_n/\partial\Delta_n)$. We will prove later that $H_n(\Delta_n/\partial\Delta_n)$ is a copy of $\mathbb{Z}$, generated by the coset $c + B_n$. The other homology groups are zero, apart from $H_0(\Delta_n/\partial\Delta_n) = \mathbb{Z}$.

One can also show that $\Delta_n/\partial\Delta_n$ is homeomorphic to S^n (and thus to $\partial\Delta_{n+1}$), so this example is consistent with the previous one. To see this, let b denote the barycentre of Δ_n, so $b_i = 1/(n+1)$ for $0 \le i \le n$. Define $m \colon \Delta_n \to \mathbb{R}$ by

$$m(t) = \max(1 - (n+1)t_0, \dots, 1 - (n+1)t_n) = (n+1)\max((b-t)_0, \dots, (b-t)_n).$$

One can check that $0 \le m(t) \le 1$, with $m(t) = 0$ iff $t = b$, and $m(t) = 1$ iff $t \in \partial\Delta_n$. Now define $f \colon \Delta_n \to B^n$ by $f(b) = 0$, and $f(t) = m(t)d(t)/\|d(t)\|$ for $t \neq b$. One can check that this is continuous at $t = b$, and that it gives a homeomorphism $f \colon \Delta_n \to B^n$, extending the homeomorphism $f \colon \partial\Delta_n \to S^{n-1}$ considered previously. It therefore induces a homeomorphism $\overline{f} \colon \Delta_n/\partial\Delta_n \to B^n/S^{n-1}$. There is also a surjective map $g \colon B^n \to S^n$ given by

$$g(u) = (4p(u)(1 - p(u))u/\|u\|, 2p(u) - 1),$$

where $p(u) = \sqrt{1 - \|u\|^2}$. For the exceptional case $u = 0$ we have $g(0) = (0, 1)$. The map g sends the whole boundary sphere S^{n-1} to the point $(0, -1)$, so it induces a map $\overline{g} \colon B^n/S^{n-1} \to S^n$. This is easily seen to be a continuous bijection from a compact space to a Hausdorff space, so it is a homeomorphism.

Example 3.22. Consider the torus $T = S^1 \times S^1$. We will construct a certain element $c \in Z_2(T)$. It turns out that $H_2(T)$ is a copy of $\mathbb{Z}$, generated by the coset $c + B_2(T)$, but we will not prove that. The construction uses the map $\eta \colon \mathbb{R} \to S^1$ given by $\eta(t) = \exp(2\pi i t)$ (so $\eta(0) = \eta(1) = 1$). We define maps $u, v \colon \Delta_2 \to T$ by

$$u(t_0, t_1, t_2) = (\eta(t_1 + t_2), \eta(t_2))$$
$$v(t_0, t_1, t_2) = (\eta(t_2), \eta(t_1 + t_2)).$$

We find that

$$\begin{aligned}
u\delta_0(t,1-t) &= u(0,t,1-t) = (1,\eta(1-t)) \\
u\delta_1(t,1-t) &= u(t,0,1-t) = (\eta(1-t),\eta(1-t)) \\
u\delta_2(t,1-t) &= u(t,1-t,0) = (\eta(1-t),1) \\
v\delta_0(t,1-t) &= v(0,t,1-t) = (\eta(1-t),1) \\
v\delta_1(t,1-t) &= v(t,0,1-t) = (\eta(1-t),\eta(1-t)) \\
v\delta_2(t,1-t) &= v(t,1-t,0) = (1,\eta(1-t)),
\end{aligned}$$

so $u\delta_0 = v\delta_2$, $u\delta_1 = v\delta_1$ and $u\delta_2 = v\delta_0$. It follows that the chain $c = [u] - [v]$ has $d(c) = 0$, as required.

Lemma 3.23. *Suppose that $a \in Z^1(X)$. Then*

(a) *For any constant path $u\colon \Delta_1 \to X$, we have $a(u) = 0$.*
(b) *If u and v are paths in X with the same endpoints, and they are homotopic relative to those endpoints, then $a(u) = a(v)$.*
(c) *If we have a path u from x to y, and a path v from y to z, and we let w denote the joined path from x to z, then $a(w) = a(u) + a(v)$.*

Proof. (a) Let u be constant with value $x \in X$ say. Let t be the constant map from Δ_2 to X with the same value. This means that $t \circ \delta_i = u$ for all i. Thus

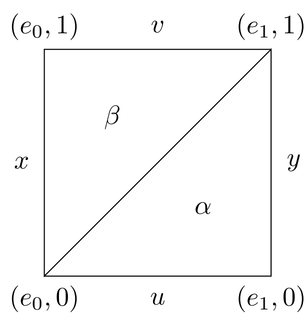
$$(\partial a)(t) = a(t \circ \delta_0) - a(t \circ \delta_1) + a(t \circ \delta_2) = a(u) - a(u) + a(u) = a(u).$$

On the other hand, we are given that $a \in Z^1(X)$, so $\partial(a) = 0$. It follows that $a(u) = 0$ as claimed.

(b) The endpoints of Δ_1 are $e_0 = (1,0)$ and $e_1 = (0,1)$. Put $x = u(e_0) = v(e_0)$ and $y = u(e_1) = v(e_1)$. The assumption is that there is a map $h\colon \Delta_1 \times [0,1] \to X$ such that $h(p,0) = u(p)$ and $h(p,1) = v(p)$ for all $p \in \Delta_1$, and $h(e_0,t) = x$ for all $t \in [0,1]$, and $h(e_1,t) = y$ for all $t \in [0,1]$. Now $\Delta_1 \times [0,1]$ is a square, which we can cut diagonally into two triangles. More explicitly, we can consider the maps $\alpha, \beta\colon \Delta_2 \to \Delta_1 \times [0,1]$ given by

$$\begin{aligned}
\alpha(t_0,t_1,t_2) &= ((t_0, t_1+t_2), t_2) \\
\beta(t_0,t_1,t_2) &= ((t_0+t_1, t_2), t_1+t_2).
\end{aligned}$$

This corresponds to the following picture:



Now put $w(t_0,t_1) = h((t_0,t_1),t_1)$, corresponding to the diagonal edge above. From the definitions and the properties of h, we find that

$$\begin{aligned}
h\alpha\delta_0(t_0,t_1) &= y & h\beta\delta_0(t_0,t_1) &= v(t_0,t_1) \\
h\alpha\delta_1(t_0,t_1) &= w(t_0,t_1) & h\beta\delta_1(t_0,t_1) &= w(t_0,t_1) \\
h\alpha\delta_2(t_0,t_1) &= u(t_0,t_1) & & \\
h\beta\delta_2(t_0,t_1) &= x. & &
\end{aligned}$$

As $a \in Z^1(X)$ we have $(\partial a)(h \circ \alpha) = (\partial a)(h \circ \beta) = 0$, so

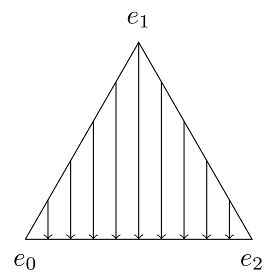
$$\begin{aligned}
a(y) - a(w) + a(u) &= 0 \\
a(v) - a(w) + a(x) &= 0
\end{aligned}$$

Here $a(x)$ and $a(y)$ refer to the constant maps $x, y\colon \Delta_1 \to X$, so $a(x) = a(y) = 0$ by part (a). We therefore have $a(u) = a(w) = a(v)$ as claimed.

(c) The path join operation is usually defined for maps $[0,1] \to X$. As we use Δ_1 instead of $[0,1]$, a little translation is required. The appropriate definition is

$$w(t_0, t_1) = \begin{cases} u(t_0 - t_1, 2t_1) & \text{if } t_0 \geq t_0 \\ v(2t_0, t_1 - t_0) & \text{if } t_0 \leq t_1. \end{cases}$$

Now define $\pi\colon \Delta_2 \to \Delta_1$ by $\pi(t_0, t_1, t_2) = (t_0 + t_1/2, t_1/2 + t_2)$. This is just a projection:



From the definitions we see that

$$\begin{aligned} w\pi\delta_0(t_0, t_1) &= w(t_0/2, t_0/2 + t_1) = v(t_0, t_1) \\ w\pi\delta_1(t_0, t_1) &= w(t_0, t_1) \\ w\pi\delta_2(t_0, t_1) &= w(t_0 + t_1/2, t_1/2) = u(t_0, t_1). \end{aligned}$$

As $a \in Z^1(X)$ we have $(\partial a)(w\pi) = 0$, so $a(v) - a(w) + a(u) = 0$ as claimed.

$\square$

Corollary 3.24. *Let X be a path conected based space. Then there is a natural isomorphism $H^1(X) \simeq \operatorname{Hom}(\pi_1(X), \mathbb{Z})$.*

Proof. Let x_0 be the basepoint in X. We can identify $\pi_1(X)$ with the set of equivalence classes of maps $u\colon \Delta_1 \to X$ for which $u\delta_0 = u\delta_1 = x_0$, under the relation of homotopy relative to endpoints. Suppose we have an element $a \in Z^1(X)$. Part (b) of the lemma tells us that there is a well-defined map $\phi(a)\colon \pi_1(X) \to \mathbb{Z}$ given by $\phi(a)([u]) = a(u)$, and part (c) tells us that $\phi(a)$ is a homomorphism. It is clear that $\phi(a+b) = \phi(a)+\phi(b)$, so we have a homomorphism $\phi\colon Z^1(X) \to \operatorname{Hom}(\pi_1(X), \mathbb{Z})$. Now suppose that $a = \partial p$ for some $p \in C^0(X)$. This means that p is a function from X to $\mathbb{Z}$, and $a(u) = p(u(e_1)) - p(u(e_0))$ for all u. In particular, if u is a based loop (so $u(e_0) = u(e_1) = x_0$) then $a(u) = 0$. From this we see that $\phi(a) = 0$, so $\phi(B^1(X)) = 0$, so there is an induced homomorphism $\overline{\phi}\colon H^1(X) \to \operatorname{Hom}(\pi_1(X), \mathbb{Z})$.

To construct a map in the opposite direction, we first choose, for each point $x \in X$, a path $\gamma_x\colon \Delta_1 \to X$ with $\gamma_x(e_0) = x_0$ and $\gamma_x(e_1) = x$. This is possible because X was assumed to be path connected. It will be convenient to insist that γ_{x_0} is just the constant path at x_0.

Next, for any $u\colon \Delta_1 \to X$ we let $\sigma(u)$ be the path formed by joining $\gamma_{u(e_0)}$, followed by u, followed by the reverse of $\gamma_{u(e_1)}$. Note that this is a based loop and so gives a class $[\sigma(u)] \in \pi_1(X)$.

Now suppose we have a homomorphism $\alpha\colon \pi_1(X) \to \mathbb{Z}$. We can define $\psi(\alpha) \in C^1(X)$ by $\psi(\alpha)(u) = \alpha([\sigma(u)])$. It is clear that $\psi(\alpha+\beta) = \psi(\alpha)+\psi(\beta)$, so we have a homomorphism $\psi\colon \operatorname{Hom}(\pi_1(X), \mathbb{Z}) \to C^1(X)$. We claim that this has $\partial\psi(\alpha) = 0$. To see this, consider a map $m\colon \Delta_2 \to X$. This gives three paths $u = m\delta_2$, $v = m\delta_0$ and $w = m\delta_1$. The map m provides a homotopy relative to endpoints between w and the join of u and v. Explicitly, the formula is

$$h((t_0, t_1), s) = \begin{cases} m(t_0 - st_1, 2st_1, t_1 - st_1) & \text{if } t_0 \geq t_1 \\ m(t_0 - st_0, 2st_0, t_1 - st_0) & \text{if } t_0 \leq t_1. \end{cases}$$

As path join is associative up to homotopy, it follows that $[\sigma(w)] = [\sigma(u)][\sigma(v)]$ in $\pi_1(X)$. As α is a homomorphism, it follows that $\alpha([\sigma(w)]) = \alpha([\sigma(u)])+\alpha([\sigma(v)])$, or equivalently $(\partial\psi(\alpha))(m) = 0$, as required. We have therefore defined a homomorphism $\psi\colon \operatorname{Hom}(\pi_1(X), \mathbb{Z}) \to Z^1(X)$, and we can compose with the quotient map $Z^1(X) \to Z^1(X)/B^1(X) = H^1(X)$ to get a homomorphism $\overline{\psi}\colon \mathrm{Hom}(\pi_1(X),\mathbb{Z}) \to H^1(X)$. We claim that this is inverse to $\overline{\phi}$.

To see this, suppose we start with $a \in Z^1(X)$ and put $\alpha = \overline{\phi}(a) \in \mathrm{Hom}(\pi_1(X),\mathbb{Z})$. Define $p \in C^0(X)$ by $p(x) = a(\gamma_x)$. If u is a path from x to y, we find using part (c) of the lemma that $\alpha([\sigma(u)]) = a(\gamma_x) + a(u) - a(\gamma_y) = (a - \partial p)(u)$. It follows that $\psi\overline{\phi}(a) = a - \partial p$, which represents the same coset in $H^1(X)$ as a does. This means that $\overline{\psi\phi} = 1\colon H^1(X) \to H^1(X)$.

In the opposite direction, suppose we start with $\alpha\colon \pi_1(X) \to \mathbb{Z}$, and put $a = \psi(\alpha) \in Z^1(X)$. Consider a based loop $u\colon \Delta_1 \to X$. Then $\phi(a)([u]) = a(u) = \psi(\alpha)(u) = \alpha([\sigma(u)])$. As γ_{x_0} was taken to be the constant path at x_0, we find that $\sigma(u)$ is homotopic (relative to endpoints) to u, so $[\sigma(u)] = [u]$, so $\phi(a)([u]) = \alpha([u])$. This means that $\overline{\phi\psi} = 1_{\mathrm{Hom}(\pi_1(X),\mathbb{Z})}$. □

4. Functoriality and homotopy invariance

We next study the effect of continuous maps on homology and cohomology groups.

Definition 4.1. Let $f\colon X \to Y$ be a continuous map.

(a) We define $f_*\colon S_k(X) \to S_k(Y)$ by $f_*(s) = f \circ s$.
(b) We then define a homomorphism $f_*\colon C_k(X) \to C_k(Y)$ by extending linearly, so $f_*(\sum_i n_i[s_i]) = \sum_i n_i[f \circ s_i]$.
(c) We will check below that this gives a chain map, so there is an induced map of homology groups, which we again denote by $f_*\colon H_*(X) \to H_*(Y)$.
(d) We also define $f^*\colon C^k(Y) \to C^k(X)$ by $(f^*v)(s) = v(f \circ s)$ (for $v \in C^k(Y) = \mathrm{Map}(S_k(Y),\mathbb{Z})$ and $s \in S_k(X)$). Equivalently, in terms of Definition 2.8 we have $f^* = D(f_*)$.
(e) We will check below that this gives a cochain map, so there is an induced map of cohomology groups, which we again denote by $f^*\colon H^*(Y) \to H_*(X)$.

Remark 4.2. Suppose we have another continuous map $g\colon Y \to Z$. It is then clear that $(gf)_* = g_*f_*$ on $S_*(X)$ and $C_*(X)$ and $H_*(X)$, and also that $(gf)^* = f^*g^*$ on $C^*(Z)$ and $H^*(Z)$. In other words, S_*, C_* and H_* are covariant functors, and C^* and H^* are contravariant functors.

We still need the following lemma to validate the above definition:

Lemma 4.3. *The map $f_*\colon C_*(X) \to C_*(Y)$ is a chain map, and the map $f^*\colon C^*(Y) \to C^*(X)$ is a cochain map.*

Proof. For $s \in S_k(X)$ we have

$$d(f_*[s]) = d([f \circ s]) = \sum_{i=0}^{k} (-1)^i [f \circ s \circ \delta_i] = f_*\left(\sum_{i=0}^{k} (-1)^i [s \circ \delta_i]\right) = f_*(d([s])),$$

so f_* is a chain map. It follows that f^* is a cochain map, either by the remarks in Definition 2.8, or by a similarly direct argument using Remark 3.11. □

The real work will be to check homotopy invariance.

Proposition 4.4. *Let f_0 and f_1 be continuous maps from X to Y. Then any homotopy from f_0 to f_1 gives rise to a chain homotopy from $(f_0)_*$ to $(f_1)_*$, and a cochain homotopy from f_0^* to f_1^*. We therefore have $(f_0)_* = (f_1)_*\colon H_*(X) \to H_*(Y)$ and $f_0^* = f_1^*\colon H^*(Y) \to H^*(X)$.*

Proof. We define maps $\sigma_i\colon \Delta_{k+1} \to [0,1] \times \Delta_k$ (for $0 \le i \le k$) by $\sigma_i(x) = (t, y)$, where $t = \sum_{j=i+1}^{k+1} x_j$ and

$$y_j = \begin{cases} x_j & \text{for } 0 \le j < i \\ x_i + x_{i+1} & \text{for } j = i \\ x_{j+1} & \text{for } i < j \le k. \end{cases}$$

For example, when $k=3$ we have

$$\begin{aligned}
\sigma_0(x_0,x_1,x_2,x_3,x_4) &= (x_1+x_2+x_3+x_4,(x_0+x_1,x_2,x_3,x_4)) \\
\sigma_1(x_0,x_1,x_2,x_3,x_4) &= (x_2+x_3+x_4,(x_0,x_1+x_2,x_3,x_4)) \\
\sigma_2(x_0,x_1,x_2,x_3,x_4) &= (x_3+x_4,(x_0,x_1,x_2+x_3,x_4)) \\
\sigma_3(x_0,x_1,x_2,x_3,x_4) &= (x_4,(x_0,x_1,x_2,x_3+x_4)).
\end{aligned}$$

It can be shown that $\Delta_k \times [0,1]$ is the union of the images of the maps σ_i, and the intersection of any two of these images is another simplex of lower dimension. However, we will not need this so we omit the proof.

Now suppose we have a homotopy $h\colon [0,1]\times X \to Y$ with $h(0,x)=f_0(x)$ and $h(1,x)=f_1(x)$. Given $u\colon \Delta_k \to X$ and $i\in\{0,\dots,k\}$ we can form the composite

$$\Delta_{k+1} \xrightarrow{\sigma_i} [0,1]\times\Delta_k \xrightarrow{1\times u} [0,1]\times X \xrightarrow{h} Y,$$

which is an element of $S_k(Y)$. We can thus define $s_k\colon C_k(X)\to C_{k+1}(Y)$ by

$$s_k([u]) = \sum_{i=0}^{k}(-1)^i[h\circ(1\times u)\circ\sigma_i].$$

We claim that these form a chain homotopy between $(f_0)_*$ and $(f_1)_*$.

To see this, put

$$\begin{aligned}
\alpha_{ij} &= (\Delta_k \xrightarrow{\delta_i} \Delta_{k+1} \xrightarrow{\sigma_j} [0,1]\times\Delta_k) && \text{for } 0\le i\le k+1,\ 0\le j\le k, \\
\beta_{ij} &= (\Delta_k \xrightarrow{\sigma_j} [0,1]\times\Delta_{k-1} \xrightarrow{1\times\delta_i} [0,1]\times\Delta_k && \text{for } 0\le i\le k,\ 0\le j\le k-1
\end{aligned}$$

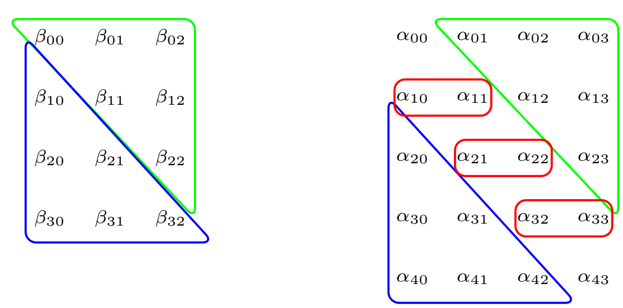
so that

$$(ds+sd)[u] = \sum_{i=0}^{k+1}\sum_{j=0}^{k}(-1)^{i+j}[h\circ(1\times u)\circ\alpha_{ij}] + \sum_{i=0}^{k}\sum_{j=0}^{k-1}(-1)^{i+j}[h\circ(1\times u)\circ\beta_{ij}].$$

One can then check from the definitions that

$$\begin{aligned}
\beta_{ij} &= \alpha_{i,j+1} && \text{for } 0\le i\le j<k \\
\beta_{ij} &= \alpha_{i+1,j} && \text{for } 0\le j<i\le k \\
\alpha_{i,i-1} &= \alpha_{ii} && \text{for } 0<i\le k.
\end{aligned}$$

Using this, we see that everything cancels to leave only the terms involving α_{00} and $\alpha_{k+1,k}$. The pattern for $k=3$ is illustrated below: the β terms in the green triangle cancel with the corresponding α terms in the other green triangle, and similarly for the blue triangles, whereas the two terms in each red rectangle cancel each other.



One can also see from the definitions that $\alpha_{00}(x)=(1,x)$ and $\alpha_{k+1,k}(x)=(0,x)$, and thus that the remaining terms are $[f_1\circ u]$ and $-[f_0\circ u]$. This means that $ds+sd=(f_1)_*-(f_0)_*$, as claimed. We thus have a chain homotopy between $(f_0)_*$ and $(f_1)_*$, which gives a cochain homotopy between f_0^* and f_1^* by the prescription $(Ds)(a)=\pm a\circ s$ as in Definition 2.8. $\square$

Definition 5.1. Let X be a space, and let Y be a subspace of X. We can then regard $C_*(Y)$ as a subcomplex of the chain complex $C_*(X)$, and thus form the quotient complex $C_*(X,Y) = C_*(X)/C_*(Y)$. We write $H_*(X,Y)$ for the homology groups of this quotient, and call these the *relative homology groups* of the pair (X,Y). We also put $C^*(X,Y) = \mathrm{Hom}(C_*(X,Y),\mathbb{Z})$ (considered as a cochain complex as in Definition 2.3) and $H^*(X,Y)$ for the corresponding cohomology groups.

Remark 5.2. Note that $C_*(\emptyset) = 0$, so $H_*(X,\emptyset) = H_*(X)$ and $H^*(X,\emptyset) = H^*(X)$. It is also clear that $H^*(X,X) = 0$.

Theorem 5.3. *Let Y be a subspace of X, and let Z be a subspace of Y. Then there are natural long exact sequences*

$$\cdots \to H_{n+1}(X,Y) \xrightarrow{\delta} H_n(Y,Z) \xrightarrow{i_*} H_n(X,Z) \xrightarrow{p_*} H_n(X,Y) \xrightarrow{\delta} H_{n-1}(Y,Z) \to \cdots$$

and

$$\cdots \to H^{n-1}(Y,Z) \xrightarrow{\delta} H^n(X,Y) \xrightarrow{p^*} H^n(X,Z) \xrightarrow{i^*} H^n(Y,Z) \xrightarrow{\delta} H^{n+1}(X,Y) \to \cdots .$$

In particular, in the case $Z = \emptyset$ we have long exact sequences

$$\cdots \to H_{n+1}(X,Y) \xrightarrow{\delta} H_n(Y) \xrightarrow{i_*} H_n(X) \xrightarrow{p_*} H_n(X,Y) \xrightarrow{\delta} H_{n-1}(Y) \to \cdots$$

and

$$\cdots \to H^{n-1}(Y) \xrightarrow{\delta} H^n(X,Y) \xrightarrow{p^*} H^n(X) \xrightarrow{i^*} H^n(Y) \xrightarrow{\delta} H^{n+1}(X,Y) \to \cdots .$$

Proof. First, it is elementary that the inclusion $C_*(Y) \to C_*(X)$ induces an inclusion $C_*(Y)/C_*(Z) \to C_*(X)/C_*(Z)$ with cokernel $C_*(X)/C_*(Y)$, so we have a short exact sequence of chain complexes

$$0 \to C_*(Y,Z) \to C_*(X,Z) \to C_*(X,Y) \to 0.$$

Proposition 2.11 therefore gives us a long exact sequence

$$\cdots \to H_{n+1}(X,Y) \xrightarrow{\delta} H_n(Y,Z) \xrightarrow{i_*} H_n(X,Z) \xrightarrow{p_*} H_n(X,Y) \xrightarrow{\delta} H_{n-1}(Y,Z) \to \cdots$$

as claimed. Next, we can apply $\mathrm{Hom}(-,\mathbb{Z})$ to the above short exact sequence to get another sequence

$$0 \to C^*(X,Y) \to C^*(X,Z) \to C^*(Y,Z) \to 0,$$

which we claim is again short exact. It is not true that the functor $\mathrm{Hom}(-,\mathbb{Z})$ automatically preserves all short exact sequences so some argument is required. In this case, however, we have $S_n(Y) \subseteq S_n(X)$, and $C_n(X,Y)$ is freely generated by $S_n(X) \setminus S_n(Y)$, so $C^n(X,Y) = \mathrm{Map}(S_n(X) \setminus S_n(Y),\mathbb{Z})$. After describing $C^n(X,Z)$ and $C^n(Y,Z)$ in the same way, and noting that

$$S_n(X) \setminus S_n(Z) = (S_n(X) \setminus S_n(Y)) \amalg (S_n(Y) \setminus S_n(Z))$$

we see that the sequence

$$0 \to C^n(X,Y) \to C^n(X,Z) \to C^n(Y,Z) \to 0,$$

can be identified with the sequence

$$0 \to C^n(X,Y) \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} C^n(X,Y) \oplus C^n(Y,Z) \xrightarrow{\left(\begin{smallmatrix}0&1\end{smallmatrix}\right)} C^n(Y,Z) \to 0,$$

so it is clearly short exact. Proposition 1.24 therefore gives us another long exact sequence

$$\cdots \to H^{n-1}(Y,Z) \xrightarrow{\delta} H^n(X,Y) \xrightarrow{p^*} H^n(X,Z) \xrightarrow{i^*} H^n(Y,Z) \xrightarrow{\delta} H^{n+1}(X,Y) \to \cdots ,$$

as claimed. ∎

Remark 5.4. In the above proof, we used an isomorphism $C_n(X,Y) \simeq \mathbb{Z}[S_n(X) \setminus S_n(Y)]$. Note that we can have a map $u\colon \Delta_n \to X$ with $u(\Delta_n) \not\subseteq Y$ but $u\delta_i(\Delta_{n-1}) \subseteq Y$ for some i (or even for all i). Because of this, the description $C_n(X,Y) \simeq \mathbb{Z}[S_n(X) \setminus S_n(Y)]$ does not interact well with the differentials, and so does not give a splitting $C_*(X) = C_*(X,Y) \oplus C_*(Y)$ of chain complexes, or a splitting $H_*(X) = H_*(X,Y) \oplus H_*(Y)$

of abelian groups. This does not matter in the above proof, because we are only using the description to check that the sequence

$$0 \to C^n(X,Y) \xrightarrow{\binom{1}{0}} C^n(X,Y) \oplus C^n(Y,Z) \xrightarrow{(0\ 1)} C^n(Y,Z) \to 0$$

is short exact, and that statement does not involve any differentials.

Definition 5.5. We define the *augmented chain complex* $\widetilde{C}_*(X)$ by

$$\widetilde{C}_n(X) = \begin{cases} C_n(X) & \text{if } n \geq 0 \\ \mathbb{Z} & \text{if } n = -1 \\ 0 & \text{if } n < -1. \end{cases}$$

The differential on $\widetilde{C}_*(X)$ is the same as for $C_*(X)$ except that $d\colon \widetilde{C}_0(X) = \mathbb{Z}[X] \to \widetilde{C}_{-1}(X) = \mathbb{Z}$ is given by $d[x] = 1$ for all x (so for $u \in S_1(X)$ we have $dd[u] = d([u(e_1)] - [u(e_0)]) = 1 - 1 = 0$ as required). We write $\widetilde{H}_*(X)$ for the homology groups of $\widetilde{C}_*(X)$, and call these the *reduced homology groups* of X.

Proposition 5.6.

(a) *If $X = \emptyset$ then $\widetilde{H}_{-1}(X) = \mathbb{Z}$ and all other reduced homology groups are zero.*
(b) *If X is contractible then $\widetilde{H}_*(X) = 0$.*
(c) *More generally, we always have $H_n(X) = \widetilde{H}_n(X)$ for all $n > 0$, and a choice of point $a \in X$ gives rise to an isomorphism $\widetilde{H}_0(X) = H_0(X,\{a\})$ and a splitting $H_0(X) = \mathbb{Z} \oplus \widetilde{H}_0(X)$.*

Proof. For claim (a), when $X = \emptyset$ and $n \geq 0$ we have $S_n(X) = \emptyset$ and $C_n(X) = 0$. It follows that the only nontrivial group in $\widetilde{C}_*(X)$ is $\widetilde{C}_{-1}(X) = \mathbb{Z}$, and the claim is clear from this. For the rest of the proof we can assume we have a point $a \in X$. We let $i\colon \{a\} \to X$ be the inclusion, and we let $r\colon X \to \{a\}$ be the constant map with value a, so $ri = 1$.

Next, it is immediate from the definitions that $\widetilde{H}_n(X) = H_n(X)$ for $n > 0$, and that $\widetilde{H}_n(X) = H_n(X) = 0$ for $n < -1$. We also have a long exact sequence

$$H_n(\{a\}) \xrightarrow{i_*} H_n(X) \xrightarrow{\pi} H_n(X,\{a\}) \xrightarrow{\delta} H_{n-1}(\{a\})$$

as in Theorem 5.3. As $ri = 1$ we have $r_* i_* = 1$ so i_* is injective so $\delta = 0$ and we actually have a short exact sequence

$$H_n(\{a\}) \xrightarrow{i_*} H_n(X) \xrightarrow{\pi} H_n(X,\{a\}).$$

For $n \neq 0$ we just have $H_n(\{a\}) = 0$ and so $H_n(X) = H_n(X,\{a\})$. For $n = 0$ it follows that the map

$$(r_*, \pi)\colon H_0(X) \to H_0(\{a\}) \oplus H_0(X,\{a\}) = \mathbb{Z} \oplus H_0(X,\{a\})$$

is an isomorphism.

Next, we can regard $\widetilde{C}_*(\{a\})$ as a subcomplex of $\widetilde{C}_*(X)$, and the quotient $\widetilde{C}_*(X)/\widetilde{C}_*(\{a\})$ is the same as $C_*(X)/C_*(\{a\}) = C_*(X,\{a\})$. The short exact sequence $\widetilde{C}_*(\{a\}) \to \widetilde{C}_*(X) \to C_*(X,\{a\})$ gives a long exact sequences of homology groups. The complex $\widetilde{C}_*(\{a\})$ has the form

$$\cdots \leftarrow 0 \leftarrow 0 \leftarrow \mathbb{Z} \xleftarrow{1} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{1} \mathbb{Z} \xleftarrow{0} \cdots$$

so $H_*(\widetilde{C}_*(\{a\})) = 0$, to the long exact sequence gives an isomorphism $\widetilde{H}_*(X) \to H_*(X,\{a\})$.

If X is contractible we have seen that $H_0(X) = \mathbb{Z}$ and $H_n(X) = 0$ for $n \neq 0$. It now follows easily that $\widetilde{H}_*(X) = H_*(X,\{a\}) = 0$. □

6. Products in cohomology

We now turn to the product structure.

Definition 6.1. Define $\lambda = \lambda_{n,m}\colon \Delta_n \to \Delta_{n+m}$ and $\rho = \rho_{n,m}\colon \Delta_m \to \Delta_{n+m}$ by

$$\lambda(x_0,\dots,x_n) = (x_0,\dots,x_n,0,\dots,0)$$
$$\rho(x_0,\dots,x_m) = (0,\dots,0,x_0,\dots,x_m).$$

Given $a \in C^n(X)$ and $b \in C^m(X)$ we then define $ab \in C^{n+m}(X)$ by

$$(ab)(w) = a(w \circ \lambda)\, b(w \circ \rho).$$

We also write 1 for the constant function $S_0(X) \to \mathbb{Z}$ with value 1, so that $1 \in C^0(X)$.

Remark 6.2. This product is sometimes called the *cup product* and written as $a \cup b$, especially in the older literature. However, we will use more streamlined notation as above.

Remark 6.3. Suppose we have a map $f\colon W \to X$, and cochains a and b as above. It is immediate from the definitions that $f^*(ab) = f^*(a)f^*(b)$.

Proposition 6.4. *The above product structure is associative, with 1 as a two-sided unit element. Moreover, we have $d(1) = 0$ and the Leibniz formula*

$$d(ab) = d(a)b + (-1)^n a\, d(b)$$

holds for all $a \in C^n(X)$ and $b \in C^m(X)$, so $C^*(X)$ becomes a differential graded ring.

Proof. First suppose that $a \in C^n(X)$ and $b \in C^0(X)$. We note that $\lambda_{n,0} = 1\colon \Delta_n \to \Delta_n$, so $(ab)(w) = a(w)b(w\rho)$. If $b = 1$ then $b(w\rho) = 1$ for any w, and so $(ab)(w) = a(w)$. This shows that $a1 = a$ for all a, and essentially the same argument shows that $1a = a$. It is also clear that $(d(1))(u) = 1(u\delta_0) - 1(u\delta_1) = 1 - 1 = 0$, so $d(1) = 0$.

Next, suppose we have $a \in C^n(X)$ and $b \in C^m(X)$ and $c \in C^p(X)$. Define $\nu = \nu_{n,m,p}\colon \Delta_m \to \Delta_{n+m+p}$ by

$$\nu(x_0, \dots, x_m) = (0, \dots, 0, x_0, \dots, x_m, 0, \dots, 0)$$

(with n zeros in the left block and p zeros in the right block). We find that $\nu = \lambda_{n+m,p}\rho_{n,m} = \rho_{n,m+p}\lambda_{m,p}$ and thus that

$$(a(bc))(w) = a(w\lambda_{n,m+p})b(w\nu_{n,m,p})c(w\rho_{n+m,p}) = ((ab)c)(w).$$

This shows that the product is associative.

Finally, we consider the claim about $d(ab)$. On the right hand side, we have

$$(d(a)b)(v) = d(a)(v\lambda)b(v\rho) = \sum_{i=0}^{n+1}(-1)^i a(v\lambda\delta_i)b(v\rho)$$

$$(-1)^n(ad(b))(v) = (-1)^n a(v\lambda)\, db(v\rho) = \sum_{j=0}^{m+1}(-1)^{n+j}a(v\lambda)b(v\rho\delta_j)$$

One can check that $\lambda\delta_{n+1} = \lambda$ and $\rho\delta_0 = \rho$, so the last term in the first sum is $(-1)^{n+1}a(v\lambda)b(v\rho)$, whereas the first term in the second sum is $(-1)^n a(v\lambda)b(v\rho)$. These terms cancel so we can drop both of them to get

$$(d(a)b + (-1)^n ad(b))(v) = \sum_{i=0}^{n}(-1)^i a(v\lambda\delta_i)b(v\rho) + \sum_{j=1}^{m+1}(-1)^{n+j}a(v\lambda)b(v\rho\delta_j).$$

On the left hand side, we note that $(d(ab))(v)$ is the sum of terms $k_i = (-1)^i(ab)(v\delta_i) = (-1)^i a(v\delta_i\lambda)b(v\delta_i\rho)$ for $0 \le i \le n+m+1$. If $0 \le i \le n$ then one can check that $\delta_i\lambda = \lambda\delta_i$ and $\delta_i\rho = \rho$ so $k_i = (-1)^i a(v\lambda\delta_i)b(v\rho)$. On the other hand, if $n < i \le n+m+1$ then we can write $i = n+j$ with $1 \le j \le m+1$. We then find that $\delta_i\lambda = \lambda$ and $\delta_i\rho = \rho\delta_j$ so $k_i = (-1)^{n+j}a(v\lambda)b(v\rho\delta_j)$. It now follows that $d(ab) = d(a)b + (-1)^n ad(b)$ as claimed. □

Corollary 6.5. *There is a well-defined unital and associative product on $H^*(X)$ given by*

$$(a + B^*(X))(b + B^*(X)) = ab + B^*(X)$$

for $a, b \in Z^*(X)$.

Proof. This follows from Proposition 1.9. □

Proposition 6.6. *The product on $H^*(X)$ satisfies the graded commutativity rule $ba = (-1)^{ij}ab$ for all $a \in H^i(X)$ and $b \in H^j(X)$.*

We will prove this by constructing some explicit chain homotopies. It is possible to avoid this by a more abstract approach (the “method of acyclic models”), but we will not discuss that here.

Definition 6.7. For $n, p \geq 0$ and $m > 0$ we define maps

$$\nu_{nmp} \colon \Delta_m \to \Delta_{n+m+p}$$
$$\mu_{nmp} \colon \Delta_{n+p+1} \to \Delta_{n+m+p}$$

by

$$\nu_{nmp}(x_0, \dots, x_m) = (0^n, x_0, \dots, x_m, 0^p)$$
$$\mu_{nmp}(y_0, \dots, y_{n+p+1}) = (y_0, \dots, y_n, 0^{m-1}, y_{n+1}, \dots, y_{n+p+1}).$$

Here 0^k denotes a string of k zeros; this means that x_0 in the first formula occurs in the same position as y_n in the second formula, and x_m in the first formula occurs in the same position as y_{n+1} in the second formula. Next, given cochains $a \in C^i(X)$ and $b \in C^j(X)$ we define $a \cup_1 b \in C^{i+j-1}(X)$ as follows. If $j = 0$ we just take $a \cup_1 b = 0$. Otherwise, there exist pairs $(n, p) \in \mathbb{N}^2$ with $j - 1 = n + p$, and we put

$$(a \cup_1 b)(u) = \sum_{j-1=n+p} (-1)^{(i+1)(j+n)+1} a(u \circ \nu_{n,i,p}) b(u \circ \mu_{n,i,p}).$$

Lemma 6.8. *For all a and b as above, we have*

$$ab - (-1)^{ij} ba = d(a) \cup_1 b + (-1)^i a \cup_1 d(b) + d(a \cup_1 b).$$

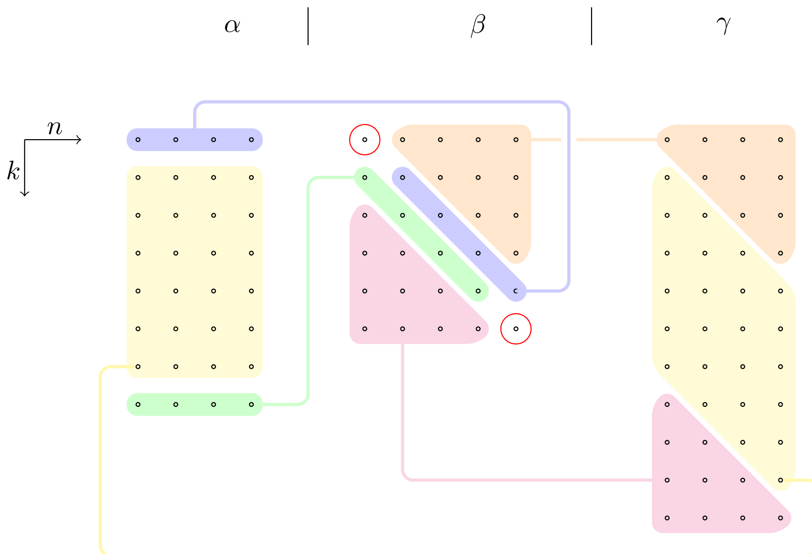
Proof. Put $c = (d(a) \cup_1 b + (-1)^i a \cup_1 d(b) + d(a \cup_1 b))(u)$; we must show that $c = (ab - (-1)^{ij} ba)(u)$. Fix an element $u \in S_{i+j}(X)$, and put

$$\begin{aligned}
\alpha_{kn} &= a(u\nu_{n,i+1,j-1-n}\delta_k) b(u\mu_{n,i+1,j-1-n}) && \text{for } 0 \leq k \leq i+1,\ 0 \leq n \leq j-1 \\
\beta_{kn} &= a(u\nu_{n,i,j-n}) b(u\mu_{n,i,j-n}\delta_k) && \text{for } 0 \leq k \leq j+1,\ 0 \leq n \leq j \\
\gamma_{kn} &= a(u\delta_k\nu_{n,i,j-1-n}) b(u\delta_k\mu_{n,i,j-1-n}) && \text{for } 0 \leq k \leq i+j,\ 0 \leq n \leq j-1.
\end{aligned}$$

From the definitions, we see that $(d(a) \cup_1 b)(u)$ is the sum of all the terms α_{kn} with suitable signs. Similarly, the terms β_{kn} give $(a \cup_1 d(b))(u)$ and the terms γ_{kn} give $d(a \cup_1 b)(u)$. We will start by analysing the combinatorics, and then make sure that the signs match up correctly at the end. The following can be verified by direct (if somewhat lengthy) inspection of the definitions:

(a) For $0 \leq n < j$ we have $\alpha_{0n} = \beta_{n+1,n+1}$.
(b) For $0 \leq n < j$ and $1 \leq k \leq i$ we have $\alpha_{kn} = \gamma_{k+n,n}$.
(c) For $0 \leq n < j$ we also have $\alpha_{i+1,n} = \beta_{n+1,n}$.
(d) $\beta_{00} = (ab)(u)$.
(e) For $0 \leq k < n \leq j$ we have $\beta_{kn} = \gamma_{k,n-1}$.
(f) The terms β_{kn} with $k = n > 0$ and $k = n + 1$ are covered by (a) and (c).
(g) For $n + 2 \leq k \leq j + 1$ (and so $n < j$) we have $\beta_{kn} = \gamma_{k+i-1,n}$.
(h) The only remaining β is $\beta_{j+1,j} = (ba)(u)$.
(i) The terms γ_{kn} with $k \leq n$ are covered by (e).
(j) The terms γ_{kn} with $n + 1 \leq k \leq n + i$ are covered by (b).
(k) The terms γ_{kn} with $n + i + 1 \leq k \leq i + j$ are covered by (g).

This pattern is displayed in the following diagram (in the case where $i = 6$, and $j = 4$).



We next consider the signs of the terms in c. If a term has sign $(-1)^e$ we will say that it has *sign exponent* e.

- For α_{kn} in $(d(a)\cup_1 b)(u)$ we have a sign exponent of $i+k$ from the definition of d and a sign exponent of $(i+2)(j+n)+1$ from the definition of $\cup_1$, giving $ij+in+i+k+1$ mod 2 in total. We will write $e_\alpha(k,n)$ for this number.
- For β_{kn} in $(-1)^i(a\cup_1 d(b))(u)$, we have a sign exponent of i from the formula, plus $(i+1)(j+n+1)+1$ from the definition of $\cup_1$, plus $j+k$ from the definition of d, making $e_\beta(k,n)=ij+in+k+n$ mod 2 in total.
- For γ_{kn} in $d(a\cup_1 b)(u)$ we have $i+j+k+1$ from the definition of d plus $(i+1)(j+n)+1$ from the definition of $\cup_1$, giving $e_\gamma(k,n)=ij+in+i+k+n$.

To make the signs work correctly in (a) to (g) above, we need the following equations in $\mathbb{Z}/2$:

(a) $e_\alpha(0,n)+e_\beta(n+1,n+1)=1$
(b) $e_\alpha(k,n)+e_\gamma(k+n,n)=1$
(c) $e_\alpha(i+1,n)+e_\beta(n+1,n)=1$
(d) $e_\beta(0,0)=0$
(e) $e_\beta(k,n)+e_\gamma(k,n-1)=1$
(g) $e_\beta(k,n)+e_\gamma(k+i-1,n)=1$
(h) $e_\beta(j+1,j)=ij+1$.

All of these can be checked easily by expanding out the definitions. □

Proof of Proposition 6.6. The statement involves elements $a\in H^i(X)$ and $b\in H^j(X)$. We choose a cocycle $\widetilde{a}\in Z^i(X)$ representing a, and a cocycle $\widetilde{b}\in Z^j(X)$ representing b. As $d(\widetilde{a})=0$ and $d(\widetilde{b})=0$, Lemma 6.8 simplifies to $\widetilde{a}\widetilde{b}-(-1)^{ij}\widetilde{b}\widetilde{a}=d(\widetilde{a}\cup_1\widetilde{b})$, so $\widetilde{a}\widetilde{b}$ represents the same cohomology class as $(-1)^{ij}\widetilde{b}\widetilde{a}$, as required. □

7. Subdivision and the Mayer-Vietoris axiom

Definition 7.1. We define $\theta_n\colon\Delta_n\to\Delta_n$ by

$$\theta_n(x)_j=\sum_{i=j}^n x_i/(i+1).$$

For example, when $n=3$ we have

$$\theta_3(x_0,x_1,x_2,x_3)=\left(x_0+\frac{x_1}{2}+\frac{x_2}{3}+\frac{x_3}{4},\frac{x_1}{2}+\frac{x_2}{3}+\frac{x_3}{4},\frac{x_2}{3}+\frac{x_3}{4},\frac{x_3}{4}\right).$$

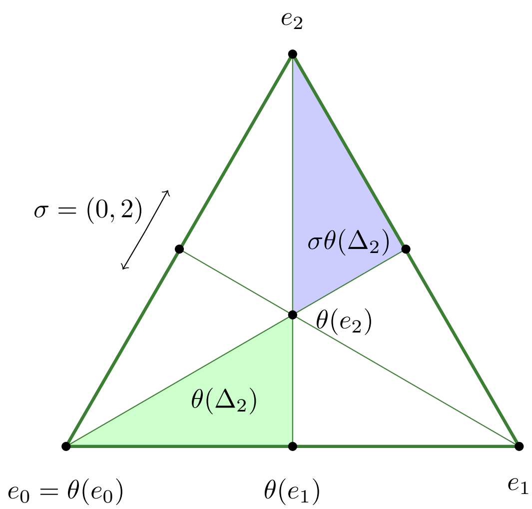
Alternatively, θ_n is the unique affine map (in the sense of Remark 3.3) such that

$$\theta_n(e_i)=(e_0+\cdots+e_i)/(i+1)$$

for all i.

Definition 7.2. We write Σ_{n+1} for the group of permutations of the set $\{0, \dots, n\}$. We let this act on the space Δ_n by the rule $(\sigma.x)_i = x_{\sigma^{-1}(i)}$ (so the effect on vertices is $\sigma.e_i = e_{\sigma(i)}$.)

The following picture shows $\theta(\Delta_2)$ and $\sigma\theta(\Delta_2)$, where σ is the transposition that exchanges 0 and 2.



It can be seen that Δ_2 is the union of the sets $\sigma\theta(\Delta_2)$, as σ runs over the six permutations of $\{0, 1, 2\}$, and that these subsets intersect nicely along the edges. This in fact holds for all n, as we see from the following lemma:

Lemma 7.3. *For all $\sigma \in \Sigma_{n+1}$ we have $x \in \sigma\theta_n(\Delta_n)$ iff*

$$x_{\sigma(0)} \geq x_{\sigma(1)} \geq \ldots \geq x_{\sigma(n)}.$$

Proof. We can easily reduce to the case where σ is the identity. If

$$x = \theta_n(y) = \left(\sum_{i=0}^{n} \frac{y_i}{i+1}, \sum_{i=1}^{n} \frac{y_i}{i+1}, \sum_{i=2}^{n} \frac{y_i}{i+1}, \dots, \frac{y_n}{n+1} \right),$$

it is immediate that $x_0 \geq x_1 \geq \cdots \geq x_n$. Conversely, given a point $x \in \Delta_n$ with $x_0 \geq x_1 \geq \cdots \geq x_n$ we can define $y_i = (i+1)(x_i - x_{i+1})$ (with the convention $x_{n+1} = 0$, so $y_n = (n+1)x_n$). We find that $y_i \geq 0$ and

$$\sum_{i=0}^{n} y_i = \left(\sum_{i=0}^{n} (i+1)x_i \right) - \left(\sum_{i=0}^{n} (i+1)x_{i+1} \right) = \left(\sum_{i=0}^{n} (i+1)x_i \right) - \left(\sum_{i=1}^{n} ix_i \right) = \sum_i x_i = 1.$$

Thus $y \in \Delta_n$. It is also clear that

$$\sum_{i=k}^{n} y_i/(i+1) = \sum_{i=k}^{n} (x_i - x_{i+1}) = x_k,$$

so $\theta_n(y) = x$ as required. $\square$

Definition 7.4. We define $\kappa_n \colon C_n(X) \to C_n(X)$ by

$$\kappa_n[u] = \sum_{\sigma \in \Sigma_{n+1}} \operatorname{sgn}(\sigma)[u \circ \sigma \circ \theta].$$

We call this the *(barycentric) subdivision* map.

Proposition 7.5. *The map $\kappa_* \colon C_*(X) \to C_*(X)$ is a chain map.*

Proof. From the definitions, we have

$$d_n\kappa_n[u] = \sum_{i=0}^{n} \sum_{\sigma \in \Sigma_{n+1}} (-1)^i \operatorname{sgn}(\sigma)[u\sigma\theta_n\delta_i].$$

Fix i with $0 \leq i < n$, and let s_i be the transposition that exchanges i and $i+1$. Put

$$A_i = \{\sigma \in \Sigma_{n+1} \mid \sigma(i) < \sigma(i+1)\},$$

so that $\Sigma_{n+1} = A_i \amalg A_i s_i$. We see from the definitions that $(\theta_n \delta_i(x))_i = (\theta_n \delta_i(x))_{i+1}$, so $s_i \theta_n \delta_i = \theta_n \delta_i$, so $\tau s_i \theta \delta_i = \tau \theta \delta_i$ for all $\tau \in A_i$. On the other hand, we have $\mathrm{sgn}(\tau s_i) = -\mathrm{sgn}(\tau)$, so the terms for τ and τs_i cancel each other. We also see directly from the definitions that $\theta_n \delta_n = \delta_n \theta_{n-1} \colon \Delta_{n-1} \to \Delta_n$. This leaves us with

$$d_n \kappa_n[u] = \sum_{\sigma \in \Sigma_{n+1}} (-1)^n \,\mathrm{sgn}(\sigma)[u\sigma\delta_n\theta_{n-1}].$$

Now put

$$B = \{(i,\tau) \mid 0 \leq i \leq n,\ \tau \in \Sigma_n\}.$$

We can define a bijection $f \colon B \to \Sigma_{n+1}$ as follows:

$$f(i,\tau)(j) = \begin{cases} \delta_i(\tau(j)) & \text{if } j < n \\ i & \text{if } j = n. \end{cases}$$

The inverse is $f^{-1}(\sigma) = (\sigma^{-1}(n), \tau)$, where

$$\tau(k) = \begin{cases} \sigma(k) & \text{if } \sigma(k) < \sigma(n) \\ \sigma(k) - 1 & \text{if } \sigma(k) > \sigma(n). \end{cases}$$

Another way to express the definition of f is as follows: we let $\tau' \in \Sigma_{n+1}$ be the obvious extension of τ with $\tau'(n) = n$, and then $f(i,\tau)$ is the composite of τ' with the cycle $(i, i+1, \dots, n)$. From this we deduce that $\mathrm{sgn}(f(i,\tau)) = (-1)^{n+i}\,\mathrm{sgn}(\tau)$. It is also clear that if $f(i,\tau) = \sigma$ then $\sigma\delta_n = \delta_i \tau$. We therefore have

$$\kappa_{n-1} d_n[u] = \sum_{i=0}^{n} \sum_{\tau \in \Sigma_n} (-1)^i \,\mathrm{sgn}(\tau)[u\delta_i\tau\theta_{n-1}] = \sum_{\sigma \in \Sigma_{n+1}} (-1)^n \,\mathrm{sgn}(\sigma)[u\sigma\delta_n\theta_{n-1}] = d_n\kappa[u]$$

as required. □

Proposition 7.6. *The map κ_* is chain-homotopic to the identity.*

We will give two proofs of this. The first is direct, but involves some rather elaborate combinatorics.

Proof. For $0 \leq k \leq n$ we define $\phi_{nk} \colon \Delta_{n+1} \to \Delta_n$ to be the unique affine map with

$$\phi_{nk}(e_i) = \begin{cases} e_i & \text{if } 0 \leq i \leq k \\ (\sum_{j<i} e_j)/i & \text{if } k < i \leq n+1. \end{cases}$$

Next, we say that a permutation $\sigma \in \Sigma_{n+1}$ is *k-monotone* if $\sigma(0) \leq \sigma(1) \leq \cdots \leq \sigma(k)$. We write $M_{n,k}$ for the set of k-monotone permutations. We define $\lambda_n \colon C_n(X) \to C_{n+1}(X)$ by

$$\lambda_n[u] = \sum_{k=0}^{n} \sum_{\sigma \in M_{n,k}} (-1)^k \,\mathrm{sgn}(\sigma)[u\sigma\phi_{n,k}].$$

We will show that $d\lambda + \lambda d = \kappa - 1$. First, we have

$$\lambda d[u] = \sum_{k=0}^{n-1} \sum_{j=0}^{n} \sum_{\tau \in M_{n-1,k}} (-1)^{j+k} \,\mathrm{sgn}(\tau)[u\delta_j\tau\phi_{n-1,k}].$$

In the proof of Proposition 7.5 we defined a bijection

$$f \colon \{0, \dots, n\} \times \Sigma_n \to \Sigma_{n+1},$$

and noted that if $\sigma = f(j,\tau)$ then $\delta_j\tau = \sigma\delta_n$ and $\mathrm{sgn}(\sigma) = (-1)^{n+j}\,\mathrm{sgn}(\tau)$. When $0 \leq k < n$, it is straightforward to check that σ is k-monotone iff τ is k-monotone, so we have a restricted bijection

$$f \colon \{0, \dots, n\} \times M_{n-1,k} \to M_{n,k}.$$

We also see from the definitions that $\delta_n\phi_{n-1,k} = \phi_{nk}\delta_{n+1}$, so $\delta_j\tau\phi_{n-1,k} = \sigma\delta_n\phi_{n-1,k} = \sigma\phi_{nk}\delta_{n+1}$. We can use this to rewrite λd as

$$\lambda d[u] = \sum_{k=0}^{n-1} \sum_{\sigma\in M_{n,k}} (-1)^{n+k} \operatorname{sgn}(\sigma)[u\sigma\phi_{nk}\delta_{n+1}].$$

Next, put

$$C = \{(i,k,\sigma) \mid 0 \leq i \leq n+1,\ 0 \leq k \leq n,\ \sigma \in M_{nk}\},$$

so

$$d\lambda[u] = \sum_{(i,k,\sigma)\in C} (-1)^{i+k} \operatorname{sgn}(\sigma)[u\sigma\phi_{nk}\delta_i].$$

When $i = n+1$ and $k = n$, the only possible σ in M_{nn} is the identity, and the corresponding term is $-[u]$. When $i = n+1$ and $k < n$ we just get the same terms as in $\lambda d[u]$ but with the opposite signs. When $i = k = 0$ we note that $M_{n0} = \Sigma_n$ and $\phi_{n0}\delta_0 = \theta_n$ so the sum of the corresponding terms is $\kappa[u]$. In summary, we see that $d\lambda[u]$ is $-\lambda d[u] + \kappa[u] - [u]$ plus some other terms indexed by the set

$$C' = \{(i,k,\sigma) \mid 0 \leq i \leq n,\ 0 \leq k \leq n,\ (i,k) \neq (0,0),\ \sigma \in M_{nk}\}.$$

We need to show that the sum over C' is zero. For this we divide C' into three parts:

$$\begin{aligned}
C'_0 &= \{(i,k,\sigma) \in C' \mid 0 \leq i \leq k\} \\
C'_1 &= \{(i,k,\sigma) \in C' \mid i = k+1\} \\
C'_2 &= \{(i,k,\sigma) \in C' \mid k+2 \leq i \leq n\}.
\end{aligned}$$

For $0 \leq i \leq k$, we let ρ_{ik} be the cyclic permutation given by

$$\rho_{nik}(m) = \begin{cases} m & \text{if } m < i \text{ or } m > k \\ m+1 & \text{if } i \leq m < k \\ i & \text{if } m = k. \end{cases}$$

Note that $\operatorname{sgn}(\rho_{ik}) = (-1)^{i+k}$, and that if σ is k-monotone, then $\sigma\rho_{ik}$ is $(k-1)$-monotone. We can thus define a map $\zeta\colon C'_0 \to C'_1$ by

$$\zeta(i,k,\sigma) = (k, k-1, \sigma\rho_{ik}).$$

One can check from the definitions that in this context we have $\sigma\phi_{nk}\delta_i = \sigma\rho_{ik}\phi_{n,k-1}\delta_k$. The first of these terms comes with the sign $(-1)^{i+k}\operatorname{sgn}(\sigma)$, whereas the second comes with the sign $(-1)^{k+k-1}\operatorname{sgn}(\rho_{ik}\sigma) = (-1)^{i+k+1}\operatorname{sgn}(\sigma)$, so the two terms cancel. Next, if $\tau \in M_{n,k-1}$ then one checks that there is a unique i such that the permutation $\sigma = \tau\rho_{ik}^{-1}$ is k-monotone. Using this we see that $\zeta\colon C'_0 \to C'_1$ is a bijection, so the terms indexed by C'_0 cancel with the terms indexed by C'_1, leaving only the terms indexed by C'_2.

Now let s_{i-1} be the transposition that exchanges $i-1$ and i as before. For $(i,k,\sigma) \in C'_2$ we put $\xi(i,k,\sigma) = (i,k,\sigma \circ s_{i-1})$. As $i \geq k+2$ we see that $\sigma \circ s_i$ is again in M_{nk}. We also see from the definitions that $\phi_{nk}(e_h)$ does not involve e_{i-1} or e_i when $h < i$, and that it involves both with the same coefficient when $h > i$. This means that $s_{i-1}\phi_{nk}\delta_i = \phi_{nk}\delta_i$, so the terms for (i,k,σ) and $\xi(i,k,\sigma)$ are the same, but with opposite signs because $\operatorname{sgn}(s_{i-1}) = -1$. It follows that the sum over C'_2 is also zero, completing the proof. □

We now give another proof, using the so-called method of acyclic models.

Definition 7.7. Suppose we are given an element $w \in C_n(\Delta_n)$. Consider a space X and an element $a = \sum_i m_i[s_i] \in C_n(X)$. Here each s_i is a map from Δ_n to X, so it induces a homomorphism $(s_i)_*\colon C_*(\Delta_n) \to C_*(X)$. We can apply this to w to get an element $(s_i)_*(w) \in C_n(X)$. We put

$$\omega(w)_X(a) = \sum_i m_i (s_i)_*(w) \in C_n(X).$$

This defines a homomorphism $\omega(w)_X\colon C_n(X) \to C_n(X)$. More generally, suppose we have a sequence $(w_0, w_1, w_2, \dots)$, with $w_n \in C_n(\Delta_n)$. We can then combine the maps $\omega(w_n)_X\colon C_n(X) \to C_n(X)$ to get a map $\omega(w)\colon C_*(X) \to C_*(X)$ of graded abelian groups.

Remark 7.8. Readers familiar with the relevant concepts will notice that the maps $\omega(w)_X$ are natural (in the sense of category theory), and that every natural map from $C_n(X)$ to $C_n(X)$ has the form $\omega(w)$ for a unique $w \in C_n(\Delta_n)$. Indeed, this follows in a straightforward way from the Yoneda Lemma.

Proposition 7.9. *Suppose we have a sequence $(w_n)_{n=0}^{\infty}$ as in Definition 7.7. Then $\omega(w)_X$ is a chain map for all X iff we have*

$$dw_n = \sum_{i=0}^{n} (-1)^i (\delta_i)_*(w_{n-1}) \in C_{n-1}(\Delta_n)$$

for all n. If so, then there is an integer r such that $\omega(w)_X$ is chain-homotopic to r times the identity map for all X. Specifically, the group $C_0(\Delta_0)$ is canonically identified with $\mathbb{Z}$, and r corresponds to w_0 under this identification.

Proof. From the definitions we have

$$\begin{aligned} d\omega(w)_X([u]) &= d(u_* w_n) = u_*(dw_n) \\ \omega(w)_X(d[u]) &= \sum_{i=0}^{n} (-1)^i \omega(w)_X([u\delta_i]) = \sum_{i=0}^{n} (-1)^i (u\delta_i)_*(w_{n-1}) = u_* \left(\sum_{i=0}^{n} (-1)^i (\delta_i)_*(w_{n-1}) \right). \end{aligned}$$

If $dw_n = \sum_i (-1)^i (\delta_i)_*(w_{n-1})$ then clearly $d \circ \omega(w)_X = \omega(w)_X \circ d$ as required. Conversely, suppose that $\omega(w)_X$ is a chain map for all X. The identity map $\iota_n \colon \Delta_n \to \Delta_n$ gives a tautological basis element $[\iota_n] \in C_n(\Delta_n)$. By considering the above calculation in the case where $X = \Delta_n$ and $u = \iota_n$, we deduce that $dw_n = \sum_i (-1)^i (\delta_i)_*(w_{n-1})$.

Now suppose instead that we start with a sequence of elements $v_n \in C_{n+1}(\Delta_n)$. In essentially the same way as before, we define $\omega(v)_X \colon C_n(X) \to C_{n+1}(X)$ by

$$\omega(v)_X(\sum_i m_i[s_i]) = \sum_i m_i \, (s_i)_*(v).$$

Using this, we define

$$\zeta(v)_X = d \circ \omega(v)_X + \omega(v)_X \circ d \colon C_*(X) \to C_*(X).$$

By construction, this is a chain map that is chain-homotopic to zero. We claim that $\zeta(v) = \omega(w)$, where w is defined by $w_0 = 0$ and

$$w_n = d(v_{n+1}) + \sum_{i=0}^{n} (-1)^i (\delta_i)_*(v_n)$$

for $n > 0$. Indeed, for $n > 0$ and $s \colon \Delta_n \to X$ we have

$$\begin{aligned} \zeta(v)_X([s]) &= d(s_*(v_{n+1})) + \omega(v)_X(\sum_{i=0}^{n} (-1)^i [s\delta_i]) \\ &= s_*(d(v_{n+1})) + \sum_{i=0}^{n} (-1)^i s_*(\delta_i)_*(v_n) \\ &= s_* \left(d(v_{n+1}) + \sum_{i=0}^{n} (-1)^i (\delta_i)_*(v_n) \right) = s_*(w_n) \end{aligned}$$

as required. The case $n = 0$ is also clear once we observe that $d \colon C_1(\Delta_0) \to C_0(\Delta_0)$ is zero.

Now suppose we have a sequence w with $w_0 = 0$ and $dw_n = \sum_{i=0}^{n} (-1)^i (\delta_i)_*(w_{n-1})$, so that $\omega(w)$ is a chain map. If we can find a sequence v as above with $w_n = d(v_{n+1}) + \sum_{i=0}^{n} (-1)^i (\delta_i)_*(v_n)$ for all $n > 0$, then the above construction will give us a nullhomotopy of $\omega(w)$. This can be done recursively, starting with

$v_0 = 0$. If $v_0, \dots, v_n$ have been found, we put $x_n = w_n - \sum_{i=0}^{n}(-1)^i(\delta_i)_*(v_n)$ and note that

$$\begin{aligned} dx_n &= dw_n - \sum_{i=0}^{n}(-1)^i(\delta_i)_*(dv_n) \\ &= \sum_{i=0}^{n}(-1)^i(\delta_i)_*(w_{n-1}) - \sum_{i=0}^{n}(-1)^i\left((\delta_i)_*(w_{n-1}) - \sum_{j=0}^{n-1}(-1)^j(\delta_j)_*(v_{n-1})\right) \\ &= -\sum_{i=0}^{n}\sum_{j=0}^{n-1}(-1)^{i+j}(\delta_i\delta_j)_*(v_{n-1}). \end{aligned}$$

By applying $\omega(v)$ to the identity $d^2(\iota_{n+1}) = 0$, we see that this is zero, so $x_n \in Z_n(\Delta_n)$. On the other hand, as Δ_n is contractible we have $H_n(\Delta_n) = 0$ and so $Z_n(\Delta_n) = B_n(\Delta_n)$, so there exists $v_{n+1} \in C_{n+1}(\Delta_n)$ with $dv_{n+1} = x_n$ as required.

More specifically, we can put $b_n = (1/(n+1), \cdots, 1/(n+1))$ (the *barycentre* of Δ_n) and define a contraction $h\colon [0,1]\times\Delta_n \to \Delta_n$ by $h(t,x) = tx+(1-t)b_n$. This gives a chain contraction of $C_*(\Delta_n)$ as in Proposition 3.17, and thus a specific choice of v_{n+1}.

This completes the proof for the case where $w_0 = 0$. For more general w we have $w_0 = d\,\iota_0$ for some $d \in \mathbb{Z}$. We can then put $w'_n = w_n - d\,\iota_n$ for all n, so $\omega(w')_X = \omega(w)_X - d.1_{C_*(X)}$. As $w'_0 = 0$ we see that $\omega(w')_X$ is chain-homotopic to zero, so $\omega(w)$ is chain-homotopic to d times the identity, as claimed. □

We can now give an alternative proof that subdivision is homotopic to the identity.

Second proof of Proposition 7.6. The map κ is $\omega(w)$, where

$$w_n = \kappa(\iota_n) = \sum_{\sigma\in\Sigma_{n+1}} \operatorname{sgn}(\sigma)[\sigma\theta_n] \in C_n(\Delta_n).$$

We have $w_0 = 1$, so it follows from Proposition 7.9 that κ is chain homotopic to 1. □

Lemma 7.10. *Let X be a compact metric space, and let U and V be open subsets with $X = U \cup V$. Then there exists $\epsilon > 0$ such that every ball of radius ϵ is either contained in U or contained in V. (Such a number ϵ is called a* Lebesgue number *for the pair (U, V).)*

Proof. For each $x \in X$ we can choose $N_x \in \{U, V\}$ such that $x \in N_x$. As N_x is open we can then find $\epsilon_x > 0$ such that the open ball $B(x, 2\epsilon_x)$ is contained in N_x. The sets $B(x, \epsilon_x)$ now cover X, so we can choose $x_1, \dots, x_m$ say so that $X = \bigcup_i B(x_i, \epsilon_{x_i})$. Put $\epsilon = \min(\epsilon_{x_1}, \dots, \epsilon_{x_m})$. Now given any ball $B(y, \epsilon)$ we can find some i such that $y \in B(x_i, \epsilon_{x_i})$ so

$$B(y,\epsilon) \subseteq B(x_i, \epsilon + \epsilon_{x_i}) \subseteq B(x_i, 2\epsilon_{x_i}) \subseteq N_{x_i} \in \{U, V\}$$

as required. □

Lemma 7.11. *If we use the metric on Δ_n given by*

$$d(x,y) = \sum_{i=0}^{n} |x_i - y_i|$$

then $d(\theta_n(x), \theta_n(y)) \leq \frac{n}{n+1} d(x,y)$.

Proof. Put $w_i = x_i - y_i$, so $\sum_i w_i = 0$. Put $\|w\| = \sum_i |w_i|$, and define a linear map $\theta_n\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by $\theta_n(x)_i = \sum_{j=i}^{n} x_j/(j+1)$. The claim is then that $\|\theta_n(w)\| \leq \frac{n}{n+1}\|w\|$. In the special case where $w = e_q - e_p$ we have $\|w\| = 2$, and we claim that $\|\theta_n(w)\| \leq 2n/(n+1)$. It will be harmless to assume that $p \leq q$, and we then have

$$\begin{aligned}
\theta(w) &= \sum_{i=0}^{q} \frac{e_i}{q+1} - \sum_{i=0}^{p} \frac{e_i}{p+1} \\
&= \sum_{i=p+1}^{q} \frac{1}{q+1} e_i - \sum_{i=0}^{p} \frac{q-p}{(p+1)(q+1)} e_i \\
\|\theta(w)\| &= \sum_{i=p+1}^{q} \frac{1}{q+1} + \sum_{i=0}^{p} \frac{q-p}{(p+1)(q+1)} = 2\frac{q-p}{q+1} \\
&= 2\left(1 - \frac{p+1}{q+1}\right) \leq 2\left(1 - \frac{1}{n+1}\right) = 2\frac{n}{n+1}
\end{aligned}$$

as required. Now consider the general case again. Put $J = \{i \mid w_i > 0\}$ and $K = \{i \mid w_i < 0\}$ and $a = \sum_{j\in J} w_j$. To avoid trivialities, we may assume that $J, K \neq \emptyset$ and so $a > 0$. As $\sum_i w_i = 0$ we also have $\sum_{k\in K} w_k = -a$ and so $d(x,y) = \sum_i |w_i| = 2a$. Now put $b_{jk} = -w_j w_k$ (for $(j,k) \in J \times K$) so $\sum_j b_{jk} = -a w_k$ and $\sum_k b_{jk} = a w_j$ and $\sum_{j,k} b_{jk} = a^2$. It is now straightforward to check that $aw = \sum_{j,k} b_{jk}(e_j - e_k)$, so

$$a\|\theta_n(w)\| \leq \sum_{j,k} b_{jk} \|\theta_n(e_j - e_k)\| \leq \sum_{j,k} b_{jk} \frac{2n}{n+1} = \frac{2na^2}{n+1}.$$

After dividing by a and recalling that $\|w\| = 2a$ we find that $\|\theta_n(w)\| \leq \frac{n}{n+1}\|w\|$ as claimed. □

Lemma 7.12. *Let U and V be open subsets of Δ_n with $\Delta_n = U \cup V$. Then for large r we have*

$$\kappa^r(\iota_n) \in C_n(U) + C_n(V) \leq C_n(\Delta_n).$$

Proof. Let ϵ be a Lebesgue number for (U, V). Then if $w \colon \Delta_n \to \Delta_n$ and the diameter of $w(\Delta_n)$ is less than ϵ, we see that either $w(\Delta_n) \subseteq U$ or $w(\Delta_n) \subseteq V$, so $[w] \in C_n(U) + C_n(V)$. It will therefore suffice to show that when r is large, the diameter of any simplex involved in $\kappa^r(\iota_n)$ is less than ϵ. The relevant simplices can all be written in the form $w = \sigma_1\theta_n\sigma_2\theta_n \cdots \sigma_n\theta_n$, where $\sigma_1, \ldots, \sigma_r$ are permutations (and so preserve distances). It follows from the previous lemma that the diameter of $w(\Delta_n)$ is at most $2(n/(n+1))^r$, which tends to zero as r increases, as required. □

Proposition 7.13. *Let Y and Z be subsets of a space X such that $X = \mathrm{int}(Y) \cup \mathrm{int}(Z)$. Then the complex $C_*(X)/(C_*(Y) + C_*(Z))$ is chain-contractible.*

Proof. Put $Q_* = C_*(X)/(C_*(Y) + C_*(Z))$. It is clear that the maps $\kappa_X \colon C_*(X) \to C_*(X)$ and $\lambda \colon C_*(X) \to C_{*+1}(X)$ preserve $C_*(Y)$ and $C_*(Z)$, so there are induced maps $\kappa \colon Q_* \to Q_*$ and $\lambda \colon Q_* \to Q_{*+1}$ with $\kappa - 1 = d\lambda + \lambda d$. We claim that for all $q \in Q_*$ there exists r such that $\kappa^r(q) = 0$. As Q_* is generated by elements $[w]$ with $w \colon \Delta_n \to X$, it will suffice to prove the claim for elements of this form. Put $U = w^{-1}(\mathrm{int}(Y))$ and $V = w^{-1}(\mathrm{int}(Z))$, so these are open subsets that cover Δ_n. It follows that for large r we have $\kappa^r(\iota_n) \in C_n(U) + C_n(V)$, so

$$\kappa^r[w] = w_*\kappa^r(\iota_n) \in w_*(C_n(U) + C_n(V)) \leq C_n(Y) + C_n(Z),$$

so $\kappa^r[w]$ represents 0 in Q_n as required.

This means that we can define $\omega \colon Q_* \to Q_*$ by $\omega(q) = \sum_{k=0}^{\infty} \kappa^k(q)$, or more explicitly $\omega(q) = \sum_{k=0}^{m} \kappa^k(q)$ for any m large enough that $\kappa^m(q) = 0$. This is a chain map which is inverse to $1 - \kappa = -d\lambda - \lambda d$. It follows that

$$d \circ (-\omega\lambda) + (-\omega\lambda) \circ d = \omega \circ (1 - \kappa) = 1,$$

so the identity on Q_* is nullhomotopic as claimed. □

Lemma 7.14. *Suppose we have a set K and subsets $I,J\subseteq K$, and let i, j, k and l be the inclusion maps as shown below:*

$$\begin{tikzcd} I\cap J \arrow[r, "i"] \arrow[d, "j"'] & I \arrow[d, "k"] \\ J \arrow[r, "l"'] & I\cup J. \end{tikzcd}$$

Then the resulting sequences

$$\mathbb{Z}[I\cap J] \xrightarrow{\binom{i_*}{j_*}} \mathbb{Z}[I]\oplus\mathbb{Z}[J] \xrightarrow{(\,k_*\ -l_*\,)} \mathbb{Z}[I\cup J]$$

and

$$\mathrm{Map}(I\cup J,\mathbb{Z}) \xrightarrow{\binom{k^*}{l^*}} \mathrm{Map}(I,\mathbb{Z})\times\mathrm{Map}(J,\mathbb{Z}) \xrightarrow{(\,i^*\ -j^*\,)} \mathrm{Map}(I\cap J,\mathbb{Z})$$

are short exact.

Proof. This is elementary and is left to the reader. □

Theorem 7.15. *Let Y and Z be subsets of a space X such that $X=\mathrm{int}(Y)\cup\mathrm{int}(Z)$, and let i, j, k and l be the inclusion maps as shown below:*

$$\begin{tikzcd} Y\cap Z \arrow[r, "i"] \arrow[d, "j"'] & Y \arrow[d, "k"] \\ Z \arrow[r, "l"'] & Y\cup Z = X \end{tikzcd}$$

Then there are natural long exact sequences as follows:

$$H_{n+1}(Y\cup Z)\xrightarrow{\delta} H_n(Y\cap Z)\xrightarrow{\binom{i_*}{j_*}} H_n(Y)\oplus H_n(Z)\xrightarrow{(\,k_*\ -l_*\,)} H_n(Y\cup Z)\xrightarrow{\delta} H_n(Y\cap Z)$$

$$H^{n-1}(Y\cap Z)\xrightarrow{\delta} H^n(Y\cup Z)\xrightarrow{\binom{k^*}{l^*}} H^n(Y)\times H^n(Z)\xrightarrow{(\,i^*\ -j^*\,)} H^n(Y\cap Z)\xrightarrow{\delta} H^{n+1}(Y\cup Z).$$

(These are called Mayer-Vietoris sequences.*) Moreover, if $a\in H^p(X)$ and $b\in H^q(Y\cap Z)$ then $\delta((ik)^*(a)b)=(-1)^p a\,\delta(b)$.*

Proof. Put $P_*=C_*(Y)+C_*(Z)\leq C_*(X)$, and $Q_*=C_*(X)/P_*$. Proposition 7.13 tells us that Q_* is chain-contractible, so $H_*(Q_*)=0$. We can thus apply Proposition 2.11 to the short exact sequence $P_*\to C_*(X)\to Q_*$, giving exact sequences

$$0=H_{n+1}(Q_*)\to H_n(P_*)\to H_n(X)\to H_n(Q_*)=0.$$

From these we see that the evident map $H_*(P_*)\to H_*(X)$ is an isomorphism. Next, we can regard $S_n(Y)$ and $S_n(Z)$ as subsets of $S_n(X)$, whose intersection is $S_n(Y\cap Z)$. The union $S_n(Y)\cup S_n(Z)$ is in general smaller than $S_n(Y\cup Z)=S_n(X)$, and we have $P_n=\mathbb{Z}[S_n(Y)\cup S_n(Z)]$ whereas $C_n(X)=\mathbb{Z}[S_n(X)]$ so $Q_n\simeq\mathbb{Z}[S_n(X)\setminus(S_n(Y)\cup S_n(Z))]$. From these descriptions we see that the dual sequence

$$D(Q_*)\to C^*(X)\to D(P_*)$$

is again short exact, and $D(Q_*)$ is cochain-contractible, so $H^*(X)$ is isomorphic to $H^*(D(P^*))$. We can also apply Lemma 7.14 to the subsets $S_n(Y),S_n(Z)\subseteq S_n(X)$ to see that the sequences

$$C_n(Y\cap Z)\xrightarrow{\binom{i_*}{j_*}} C_n(Y)\oplus C_n(Z)\xrightarrow{(\,k_*\ -l_*\,)} P_n.$$

and

$$D(P_*)_n\xrightarrow{\binom{k^*}{l^*}} C^n(Y)\times C^n(Z)\xrightarrow{(\,i^*\ -j^*\,)} C^n(Y\cap Z)$$

are short exact. We can thus use Proposition 1.24 and 2.11 to produce long exact sequences as claimed. The final claim about the multiplicative structure follows from Proposition 1.34. □

8. Tensor products of (co)chain complexes

Definition 8.1. Let U_* and V_* be graded abelian groups. We define a graded abelian group $U_* \otimes V_*$ by

$$(U_* \otimes V_*)_n = \bigoplus_{i+j=n} U_i \otimes V_j.$$

We define an isomorphism $\tau\colon U_* \otimes V_* \to V_* \otimes U_*$ by

$$\tau(u \otimes v) = (-1)^{ij} v \otimes u$$

for $u \in U_i$ and $v \in V_j$. Next, if U_* and V_* are chain complexes we define a differential on $U_* \otimes V_*$ by

$$d(u \otimes v) = d(u) \otimes v + (-1)^i u \otimes d(v).$$

Proposition 8.2. *There are natural maps*

$$\mu\colon H_*(U_*) \otimes H_*(V_*) \to H_*(U_* \otimes V_*)$$

given by

$$\mu((u + B_i(U_*)) \otimes (v + B_i(V_*))) = u \otimes v + B_i(U_* \otimes V_*) \in H_{i+j}(U_* \otimes V_*)$$

for all $u \in Z_i(U_*)$ and $v \in Z_j(V_*)$.

Proof. First, if $u \in Z_i(U_*)$ and $v \in Z_j(V_*)$ we have

$$d(u \otimes v) = 0 \otimes v + (-1)^i u \otimes 0 = 0,$$

so $u \otimes v \in Z_{i+j}(U_* \otimes V_*)$. We can thus define a bilinear map

$$\mu''\colon Z_i(U_*) \times Z_j(V_*) \to H_{i+j}(U_* \otimes V_*)$$

by

$$\mu''(u, v) = u \otimes v + B_i(U_* \otimes V_*).$$

Next, it is clear that

$$\begin{aligned}\mu''(u + d(r), v + d(s)) &= u \otimes v + d(r \otimes (v + d(s)) + (-1)^i u \otimes s) + B_i(U_* \otimes V_*) \\ &= u \otimes v + B_i(U_* \otimes V_*) = \mu''(u, v),\end{aligned}$$

so there is a well-defined bilinear map

$$\mu'\colon H_i(U_*) \times H_j(V_*) \to H_{i+j}(U_* \otimes V_*)$$

given by

$$\mu'(u + B_i(U_*), v + B_i(V_*)) = \mu''(u, v).$$

By the universal property of tensor products, this gives a homomorphism μ as described. □

Proposition 8.3. *Let q denote the evident composite*

$$Z_*(U_* \otimes V_*) \to H_*(U_* \otimes V_*) \to \mathrm{cok}(\mu)_*.$$

Then there is a unique map

$$\nu\colon \mathrm{Tor}(H_i(U_*), H_j(V_*)) \to \mathrm{cok}(\mu)_{i+j+1}$$

such that

$$\nu(e_p(u + B_i(U_*), v + B_j(V_*))) = q(r \otimes v - (-1)^i u \otimes s)$$

whenever $u \in Z_i(U_*)$, $v \in Z_j(V_*)$, $d(r) = pu$ and $d(s) = pv$.

Proof. For u, v, r and s as above, we put

$$\nu''_p(u, v, r, s) = r \otimes v - (-1)^i u \otimes s \in (U_* \otimes V_*)_{i+j+1}.$$

We have

$$d(\nu''_p(u, v, r, s)) = d(r) \otimes v - u \otimes d(s) = pu \otimes v - u \otimes pv = 0,$$

so $\nu''_p(u, v, r, s) \in Z_{i+j+1}(U_* \otimes V_*)$ and so $q(\nu''(u, v, r, s))$ is defined. Now suppose we have a different pair of elements r', s' with $d(r') = pu$ and $d(s') = pv$. This means that $r' = r + x$ and $s' = s + y$ for some $x \in Z_{i+1}(U_*)$ and $y \in Z_{j+1}(V_*)$. It follows that

$$x \otimes v + B_{i+j+1}(U_* \otimes V_*) = \mu((x + B_{i+1}(U_*)) \otimes (v + B_j(V_*))) \in \mathrm{image}(\mu),$$

so $q(x \otimes v) = 0$. Similarly, we have $q(u \otimes y) = 0$ and so

$$q(\nu''_p(u, v, r', s')) = q(\nu''_p(u, v, r, s)).$$

One can also check that

$$\nu''_p(u + d(a), v + d(b), r + pa, s + pb) = \nu''_p(u, v, r, s) - (-1)^i d(pa \otimes b + r \otimes b + a \otimes s),$$

so that

$$q(\nu''_p(u + d(a), v + d(b), r + pa, s + pb)) = q(\nu''_p(u, v, r, s)).$$

From this it follows that there is a well-defined map

$$\nu'_p \colon H_i(U_*)[p] \times H_j(V_*)[p] \to \operatorname{cok}(\mu)_{i+j+1}$$

given by

$$\nu'_p(u + B_i(U_*), v + B_j(V_*)) = q(\nu''_p(u, v, r, s))$$

for any choice of r and s as above. This is clearly bilinear. Now suppose that u represents an element of $H_i(U_*)[nk]$ and v represents an element of $H_j(V_*)[mk]$, so we can choose $r \in U_{i+1}$ and $s \in V_{j+1}$ with $d(r) = nku$ and $d(s) = mkv$. We find that

$$\begin{aligned}
\nu'_{nd}(u + B_i, mv + B_j) &= \nu''_{nd}(u, mv, r, ns) = r \otimes mv - (-1)^i u \otimes ns \\
&= mr \otimes v - (-1)^i nu \otimes s = \nu''_{md}(nu, v, mr, s) \\
&= \nu'_{md}(nu + B_i, v + B_j).
\end{aligned}$$

It follows using Proposition 7.34 in the abelian group theory notes that there is a unique homomorphism

$$\nu \colon \operatorname{Tor}(H_i(U_*), H_j(V_*)) \to H_{i+j+1}(U_* \otimes V_*)$$

with $\nu(e_p(a, b)) = \nu'_p(a, b)$ for all $a \in H_i(U_*)[p]$ and $b \in H_j(V_*)[p]$, as required. $\square$

Lemma 8.4. *Let U_* be a chain complex of free abelian groups, and put $Z_n = \ker(d \colon U_n \to U_{n-1})$ as usual. Then one can choose subgroups $T_n \leq U_n$ such that $U_n = T_n \oplus Z_n$ and the map $d \colon U_n \to U_{n-1}$ restricts to give an isomorphism $T_n \to B_{n-1}$.*

Proof. The group B_{n-1} is a subgroup of the free abelian group U_{n-1}, so it is itself a free abelian group. We can therefore choose a basis $(e_i)_{i \in I}$ for B_{n-1}. As $B_{n-1} = \operatorname{img}(d \colon U_n \to U_{n-1})$, we can choose elements $e'_i \in U_n$ with $d(e'_i) = e_i$. We can then define $s \colon B_{n-1} \to U_n$ by $s(e_i) = e'_i$, and we find that $ds = 1$. We define $T_n = s(B_{n-1}) \leq U_n$. Given $u \in U_n$ we have $d(u) \in B_{n-1}$ and so $sd(u) \in T_n$. As $ds = 1$ we have $d(u - sd(u)) = 0$, so $u - sd(u) \in Z_n$. This proves that $U_n = T_n + Z_n$. Moreover, if $u \in T_n \cap Z_n$ then $u = s(b)$ for some $b \in B_{n-1}$, but also $du = 0$. Using $ds = 1$ we get $b = 0$ and so $u = s(b) = 0$. This shows that $U_n = T_n \oplus Z_n$. It is clear by construction that d gives an isomorphism $T_n \to B_{n-1}$ with inverse s. $\square$

d

Theorem 8.5. *Let U_* and V_* be chain complexes of free abelian groups. Then the map $\nu \colon \operatorname{Tor}(H_*(U_*), H_*(V_*)) \to \operatorname{cok}(\mu)$ is an isomorphism, so we have short exact sequences*

$$(H_*(U_*) \otimes H_*(V_*))_n \xrightarrow{\mu} H_n(U_* \otimes V_*) \twoheadrightarrow \operatorname{Tor}(H_*(U_*), H_*(V_*))_{n-1}.$$

Moreover, these are split, but there is no natural choice of splitting.

Proof. First choose splittings $U_n = T_n \oplus Z_n$ as in Lemma 8.4. Here T_n and Z_n are subgroups of the free abelian group U_n, so they are also free. We can regard Z_* as a subcomplex of U_* (with trivial differential) and the quotient U_*/Z_* can be identified with T_* (again with trivial differential). The short exact sequence $Z_* \rightarrowtail U_* \twoheadrightarrow T_*$ gives a sequence $Z_* \otimes V_* \rightarrowtail U_* \otimes V_* \twoheadrightarrow T_* \otimes V_*$, which is again short exact because V_* is free. Proposition 2.11 therefore gives an exact sequence

$$H_{i+1}(T_* \otimes V_*) \xrightarrow{\delta} H_i(Z_* \otimes V_*) \xrightarrow{\alpha} H_i(U_* \otimes V_*) \xrightarrow{\beta} H_i(T_* \otimes V_*) \xrightarrow{\delta} H_{i-1}(Z_* \otimes V_*).$$

and thus a short exact sequence

$$\operatorname{cok}(\delta) \xrightarrow{\overline{\alpha}} H_*(U_* \otimes V_*) \xrightarrow{\overline{\beta}} \ker(\delta).$$

As T_* is free with trivial differential we see that $H_*(T_* \otimes V_*) = T_* \otimes H_*(V_*)$, and similarly $H_*(Z_* \otimes V_*) = Z_* \otimes H_*(V_*)$. One can check that with these identifications, the connecting map δ is just $d \otimes 1$. We can tensor the short exact sequence $T_{*+1} \xrightarrow{d} Z_* \to H_*(U)$ by $H_*(V)$ and use the freeness of T_* and Z_* again to get a four term exact sequence

$$\operatorname{Tor}(H_*(U), H_*(V))_{i+1} \rightarrowtail (T_* \otimes H_*(V))_{i+1} \xrightarrow{d\otimes 1} (Z_* \otimes H_*(V))_i \twoheadrightarrow (H_*(U_*) \otimes H_*(V_*))_i,$$

so $\operatorname{cok}(\delta) = H_*(U_*) \otimes H_*(V_*)$ and $\ker(\delta) = \operatorname{Tor}(H_*(U_*), H_*(V_*))$. This gives a short exact sequence

$$(H_*(U_*) \otimes H_*(V_*))_n \xrightarrow{\overline{\alpha}} H_n(U_* \otimes V_*) \xrightarrow{\overline{\beta}} \operatorname{Tor}(H_*(U_*), H_*(V_*))_{n-1}.$$

By inspection of the definitions, we have $\overline{\alpha} = \mu$. To understand the map $\overline{\beta}$, we use part (d) of Proposition 7.30 in the abelian group theory notes to describe the isomorphism between $\operatorname{Tor}(H_i(U_*), H_j(V_*))$ and $\ker(\delta)$. If we have an element $e_p(a, b)$ in the Tor group, we can find $u \in Z_i$ representing a and $r \in T_{i+1}$ with $d(r) = pu$, and the result mentioned tells us that the corresponding element of $\ker(\delta)$ is $r \otimes b$. Now choose $v \in V_j$ representing b and $s \in V_{j+1}$ with $d(s) = pv$, then put

$$w = \nu''_p(u, v, r, s) = r \otimes v - (-1)^i u \otimes s \in (U_* \otimes V_*)_{i+j+1}.$$

We find that w is a cycle and that the corresponding homology class maps to $s \otimes b$ in $T_* \otimes H_*(V_*)$. This proves that the map $\operatorname{cok}(\mu) \to \operatorname{Tor}(H_*(U_*), H_*(V_*))$ induced by $\overline{\beta}$ is inverse to ν. $\square$