MAGIC064 Algebraic Topology

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Some theorems

- If $n \neq m$ then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$.
- Put

$$\begin{aligned} SO_3 &= \{3 \times 3\text{rotation matrices}\} = \{A \in M_3(\mathbb{R}) \mid AA^T = I,\ \det(A) = 1\} \\ P &= \{\text{trace 1 projectors in } \mathbb{R}^4\} = \{A \in M_4(\mathbb{R}) \mid A^T = A^2 = A,\ \operatorname{trace}(A) = 1\} \\ S^3 &= \text{the 3-sphere} = \{x \in \mathbb{R}^4 \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\} \\ \mathbb{R}P^3 &= S^3/\sim \qquad \text{where } u \sim v \text{ iff } v = \pm u \end{aligned}$$

  Then SO_3, P and $\mathbb{R}P^3$ are homeomorphic to each other but not to S^3.
- The Fundamental Theorem of Algebra: if $f(z) \in \mathbb{C}[z]$ is a nonconstant polynomial, then it has a root.
- The Brouwer Fixed Point Theorem: if $f\colon [0,1]^n \to [0,1]^n$ is continuous, then there is a fixed point $x \in [0,1]^n$ with $f(x) = x$.
- The Borsuk-Ulam Theorem: if $n > m$ then there is no continuous map $f\colon S^n \to S^m$ with $f(-x) = -f(x)$ for all $x \in S^n$.

A key method for proving these results is the theory of cohomology rings.

Cohomology

For each space X, one can define a cohomology ring $H^*(X)$. For the moment, we will just list some properties of these rings (Eilenberg-Steenrod axioms).

1. $H^*(X)$ is a graded ring. For each integer $j \geq 0$ we have an abelian group $H^j(X)$, and any pair of elements $a \in H^j(X)$ and $b \in H^k(X)$ have a product $ab \in H^{j+k}(X)$. This multiplication is associative and distributes over addition. It is commutative in the graded sense: $ba = (-1)^{jk}ab$. (We also put $H^j(X) = 0$ for $j < 0$.)
2. $H^*(X)$ is contravariantly functorial in X: for any continuous map $f\colon X \to Y$ we have a ring homomorphism $f^*\colon H^*(Y) \to H^*(X)$. If we have another map $g\colon Y \to Z$ then $(gf)^* = f^*g^*\colon H^*(Z) \to H^*(X)$; also $1_X^* = 1_{H^*(X)}$.
3. $H^*(X)$ is homotopy invariant: given a continuous family of maps $f_t\colon X \to Y$ (for $0 \leq t \leq 1$) we have $f_0^* = f_1^*\colon H^*(Y) \to H^*(X)$. (The family is a *homotopy*; the maps f_0 and f_1 are *homotopic*.)
4. $H^*(\text{point}) = H^0(\text{point}) = \mathbb{Z}$.
5. Excision and Mayer-Vietoris axioms (explained later).

These properties characterise $H^*(X)$ uniquely.

Examples of cohomology rings

Example

$H^*(S^n)$ is the free abelian group generated by $1 \in H^0(S^n)$ and an element $u_n \in H^n(S^n)$. The ring structure is given by $u_n^2 = 0$ (if $n > 0$).

Example

Suppose we have distinct points $a_1, \dots, a_n \in \mathbb{C}$ and put $M = \mathbb{C} \setminus \{a_1, \dots, a_n\}$. Define $f_i\colon M \to S^1$ by $f_i(z) = (z - a_i)/|z - a_i|$ and put $v_i = f_i^*(u_1)$. Then $H^*(M)$ is the free abelian group generated by $1 \in H^0M$ and $v_1, \dots, v_m \in H^1M$. The ring structure is given by $v_iv_j = 0$ for all i, j.

Example

Put $F_n\mathbb{C} = \{z \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}$. Given $i \neq j \in \{1, \dots, n\}$ we define $f_{ij}\colon F_n\mathbb{C} \to S^1$ by $f_{ij}(z) = (z_i - z_j)/|z_i - z_j|$, and put $a_{ij} = f_{ij}^*u_1$. Then $H^*(F_n\mathbb{C})$ is freely generated by the elements a_{ij} modulo relations $a_{ij} = a_{ji}$ and $a_{ij}^2 = 0$ and $a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij} = 0$ for all i, j, k. One can also give a basis for this ring as a free abelian group.

The group $H^0(X)$

- The points of a space X can be grouped into path components, where x and y lie in the same path component iff there is a continuous path $s\colon [0,1] \to X$ with $s(0) = x$ and $s(1) = y$.
- We write $\pi_0(X)$ for the set of path components in X.
- We write $\mathrm{Map}(\pi_0(X), \mathbb{Z})$ for the set of functions from $\pi_0(X)$ to $\mathbb{Z}$. This is a ring under pointwise addition and multiplication. For example, if X has three path components, then $\mathrm{Map}(\pi_0(X), \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.
- It works out that $H^0(X)$ is always isomorphic to $\mathrm{Map}(\pi_0(X), \mathbb{Z})$. For example, in the common case where X is path-connected, we just have $H^0(X) = \mathbb{Z}$.
- If X has the discrete topology, then $\pi_0(X) = X$ so $H^0(X) = \mathrm{Map}(X, \mathbb{Z})$. In this case it can be shown that $H^n(X) = 0$ for all $n \neq 0$.

Steps to define $H^*(X)$

We will

- Define what we mean by a *cochain complex*
- Define what we mean by a *graded ring*
- Define what we mean by a *differential graded ring*: a cochain complex with compatible graded ring structure
- Define the cohomology of a cochain complex, and show that the cohomology of a DGR is a graded ring
- For each topological space X define a differential graded ring $C^*(X)$, called the singular cochain complex of X
- Define $H^*(X)$ to be the cohomology of $C^*(X)$.

Cochain complexes and differential graded rings

- A *cochain complex* is a system of abelian groups U^i (for $i \in \mathbb{Z}$) equipped with homomorphisms $d\colon U^i \to U^{i+1}$ such that each composite

$$U^{i-1} \xrightarrow{d} U^i \xrightarrow{d} U^{i+1}$$

  is zero (or more briefly, $d^2 = 0$).
  In almost all cases we will have $U^i = 0$ for $i < 0$.
- A *differential graded ring* is a cochain complex A^* together with an element $1 \in A^0$ and a multiplication rule giving $ab \in A^{i+j}$ for all $a \in A^i$ and $b \in A^j$, such that:

$$\begin{aligned}
1a = a = a1 &\text{ for all } a \in A^i \\
a(bc) = (ab)c &\text{ for all } a \in A^i,\ b \in A^j,\ c \in A^k \\
a(b+c) = ab + ac &\text{ for all } a \in A^i,\ b, c \in A^j \\
(a+b)c = ac + bc &\text{ for all } a, b \in A^i,\ c \in A^j \\
d(1) = 0 & \\
d(ab) = d(a)b + (-1)^i a\, d(b) &\text{ for all } a \in A^i,\ b \in A^j.
\end{aligned}$$

  The last relation is called the *Leibniz rule*.

Cohomology of a cochain complex

- Let A^* be a cochain complex. For $i \in \mathbb{Z}$ we put

$$Z^i(A^*) = \ker(d\colon A^i \to A^{i+1}) \leq A^i \qquad \text{(group of cocycles)}$$
$$B^i(A^*) = \mathrm{img}(d\colon A^{i-1} \to A^i) \leq A^i \qquad \text{(group of coboundaries)}$$

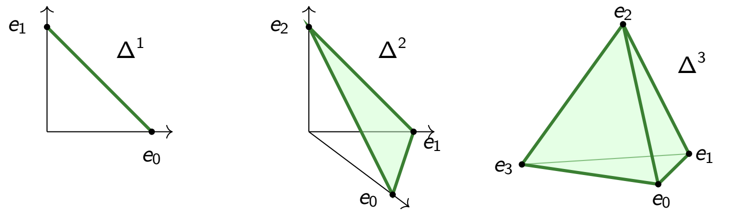
- As $d^2 = 0$ we have $d(B^i(A^*)) = 0$ and so $B^i(A^*) \leq Z^i(A^*)$.
  It is therefore meaningful to define $H^i(A^*) = Z^i(A^*)/B^i(A^*)$.
  Elements of $H^i(X)$ are cosets $[z] = z + B^i(X)$, called *cohomology classes*.
- If A^* is clear from the context, we will just write Z^i, B^i and H^i instead of $Z^i(A^*)$, $B^i(A^*)$ and $H^i(A^*)$.
- We write Z^* for the sequence of groups Z^i, and similarly for B^* and H^*.
- Now let A^* be a DGR. Using the Leibniz rule $d(ab) = d(a)b \pm a\, d(b)$ we find that Z^* is a subring of A^* and that B^* is an ideal in Z^*.
- It follows that $H^*(A^*)$ has an induced ring structure with $[z][w] = [zw]$ for $z \in Z^n$ and $w \in Z^m$.
- Example: $A^* = \mathbb{Z}[x] \oplus \mathbb{Z}[x]a$ with $d(a) = x$ so $d(x^n) = 0$, $d(x^n a) = x^{n+1}$.

$$1 \xrightarrow{0} a \longmapsto x \xrightarrow{0} xa \longmapsto x^2 \xrightarrow{0} x^2 a \longmapsto x^3$$

  $Z^{2k+1} = B^{2k+1} = 0$ and $Z^{2k} = B^{2k} = \mathbb{Z}x^k$ except $Z^0 = \mathbb{Z}$ and $B^0 = 0$.
  Thus $H^0(A^*) = \mathbb{Z}$ and $H^n(A^*) = 0$ for $n \neq 0$.

Simplices

- The *standard n-simplex* is the space
$$\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ for all } i \text{ and } \sum_i x_i = 1\}.$$
The *vertices* of Δ_n are just the standard basis vectors $e_0, \dots, e_n$, so $e_0 = (1, 0, \dots, 0)$ and $e_1 = (0, 1, 0, \dots, 0)$ and $e_n = (0, \dots, 0, 1)$.
- Δ^0 is a point, Δ^1 is an interval, Δ^2 is a triangle, Δ^3 is a tetrahedron.



- We always identify $(1-t, t) \in \Delta^1$ with $t \in [0, 1]$, so $e_0 \sim 0$ and $e_1 \sim 1$.
- We define $S_k(X) = \mathrm{Cont}(\Delta^k, X)$, the set of continuous maps $\Delta^k \to X$.
As $\Delta^0 =$ point we can identify $S_0(X)$ with X.
As $\Delta^1 = [0, 1]$ we can identify $S_1(X)$ with the set of paths in X.
Loosely: $S_2(X)$ is the set of triangles in X.

Singular cochains

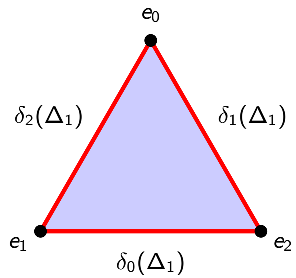
- Define $C^k(X) = \mathrm{Map}(S_k(X), \mathbb{Z})$ (the set of all functions from $S_k(X)$ to $\mathbb{Z}$).
- $S_0(X) = X$ so $C^0(X) = \mathrm{Map}(X, \mathbb{Z})$
(the set of all maps $X \to \mathbb{Z}$, no continuity requirement)
(This is a commutative ring under pointwise addition and multiplication)
- $S_1(X)$ is the set of paths in X, so $C^1(X)$ is the set of functions from paths to integers.
- We define $d\colon C^0(X) \to C^1(X)$ by $d(f)(u) = f(u(1)) - f(u(0))$ for $f \in C^0(X)$ and $u\colon [0, 1] \to X$.
- More detail:
  - $f \in C^0(X)$ so $f\colon X \to \mathbb{Z}$.
  - We need to define $d(f) \in C^1(X) = \mathrm{Map}(S_1(X), \mathbb{Z})$, so for $u \in S_1(X)$ we need to define $d(f)(u) \in \mathbb{Z}$.
  - Here $u\colon [0, 1] \to X$ so $u(0), u(1) \in X$.
  - As $f\colon X \to \mathbb{Z}$ we have $f(u(0)), f(u(1)) \in \mathbb{Z}$.
  - We put $d(f)(u) = f(u(1)) - f(u(0))$.
- For $k < 0$ we define $S_k(X) = \emptyset$ and $C^k(X) = 0$ and $d = 0\colon C^k(X) \to C^{k+1}(X)$.
- We will define $d\colon C^k(X) \to C^{k+1}(X)$ for $k > 0$ later.

Zeroth cohomology

- $H^0(X) = Z^0(X)/B^0(X)$.
- $B^0(X) = \mathrm{img}(d = 0\colon C^{-1}(X) = 0 \to C^0(X))$,
so $B^0(X) = 0$, so $H^0(X) = Z^0(X)$.
- $Z^0(X) = \ker(d\colon C^0(X) \to C^1(X)) = \{f \in \mathrm{Map}(X, \mathbb{Z}) \mid d(f) = 0\}$.
- For a path $u\colon [0, 1] \to X$ we have $d(f)(u) = f(u(1)) - f(u(0))$,
so $d(f) = 0$ iff $f(u(1)) = f(u(0))$ for all paths u.
- In other words, $H^0(X) = Z^0(X)$ is the set of maps $f\colon X \to \mathbb{Z}$ such that $f(x) = f(y)$ whenever x and y can be connected by a path in X.
- In other words, $H^0(X)$ is the set of maps $f\colon X \to \mathbb{Z}$ that are constant on each path component.
- Thus, if $\pi_0(X)$ is the set of path components, then
$H^0(X) = \mathrm{Map}(\pi_0(X), \mathbb{Z})$.
- X is *path connected* if it is nonempty and any two points can be joined by a path. If so, then $|\pi_0(X)| = 1$ and $H^0(X)$ is just the set of constant functions $X \to \mathbb{Z}$ so $H^0(X) \simeq \mathbb{Z}$.

Face maps

- For $0 \leq i \leq n$ we define $\delta_i\colon \Delta_{n-1} \to \Delta_n$ by inserting 0 in position i:
$$\delta_i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1}).$$
- This is the inclusion of the face opposite e_i.
- For $n = 2$:
$$\delta_0(t_0, t_1) = (0, t_0, t_1) \qquad \delta_1(t_0, t_1) = (t_0, 0, t_1) \qquad \delta_2(t_0, t_1) = (t_0, t_1, 0).$$



- For $n = 1$: the maps $\delta_0, \delta_1\colon \Delta^0 = \{e_0\} \to \Delta^1$ are given by $\delta_0(e_0) = e_1$ and $\delta_1(e_0) = e_0$.

The differential

We define $d\colon C^k(X) \to C^{k+1}(X)$ by

$$d(f)(v) = \sum_{i=0}^{k+1} (-1)^i f(v \circ \delta_i).$$

In more detail:

- f is assumed to be an element of the group $C^k(X) = \mathrm{Map}(S_k(X), \mathbb{Z})$, so for each $u \in S_k(X)$ we have an integer $f(u)$.
- $d(f)$ is supposed to be an element of the group $C^{k+1}(X) = \mathrm{Map}(S_{k+1}(X), \mathbb{Z})$, so for each element $v \in S_{k+1}(X)$ we need to define the element $d(f)(v) \in \mathbb{Z}$.
- So suppose we have $v \in S_{k+1}(X)$, i.e. v is a continuous map $\Delta^{k+1} \to X$. For $0 \le i \le k+1$ we have a face map $\delta_i\colon \Delta_k \to \Delta_{k+1}$, which we can compose with v to get a continuous map $v \circ \delta_i\colon \Delta^k \to X$, or in other words an element $v \circ \delta_i \in S_k(X)$.
  As $f\colon S_k(X) \to \mathbb{Z}$, we therefore have an integer $f(v \circ \delta_i) \in \mathbb{Z}$.
- We define $d(f)(v)$ to be the alternating sum of the above integers, i.e. $d(f)(v) = \sum_{i=0}^{k+1} (-1)^i f(v \circ \delta_i)$.

The face relation

- Claim: If $0 \le j \le i \le k$ then $\delta_j \delta_i = \delta_{i+1} \delta_j\colon \Delta^{k-1} \to \Delta^k \to \Delta^{k+1}$.
- Example: $\delta_2 \delta_3 = \delta_4 \delta_2\colon \Delta^2 \to \Delta^4$:

$$\begin{array}{rlccccccc}
 & & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\delta_2(t) = & (& t_0, & t_1, & 0, & t_2, & t_3, & t_4) & \\
\delta_4(\delta_2(t)) = & (& t_0, & t_1, & 0, & t_2, & 0, & t_3, & t_4) \\
\delta_3(t) = & (& t_0, & t_1, & t_2, & 0, & t_3, & t_4) & \\
\delta_2(\delta_3(t)) = & (& t_0, & t_1, & 0, & t_2, & 0, & t_3, & t_4)
\end{array}$$

- Claim: the composite $C^{k-1}(X) \xrightarrow{d} C^k(X) \xrightarrow{d} C^{k+1}(X)$ is zero.
- By definition, for $f \in C^{k-1}(X)$ and $u \in S_{k+1}(X)$ we have

$$d^2(f)(u) = \sum_{j=0}^{k+1} (-1)^i d(f)(u\delta_j) = \sum_{j=0}^{k+1} \sum_{i=0}^{k} (-1)^{i+j} f(u\delta_j\delta_i).$$

  The relation $\delta_j \delta_i = \delta_{i+1} \delta_j$ shows that some terms are the same.
  The $+1$ ensures that matching terms have opposite signs and so cancel.
  With more care we can see that there is nothing left, so $d^2(f)(u) = 0$.
- Thus: $C^*(X)$ is a cochain complex, and we can define $Z^k(X) = \ker(d\colon C^k(X) \to C^{k+1}(X))$ and $B^k(X) = \mathrm{img}(d\colon C^{k-1}(X) \to C^k(X))$ and $H^k(X) = Z^k(X)/B^k(X)$.

Cohomology of discrete spaces

- Claim: if X is discrete then $H^0(X) = \mathrm{Map}(X, \mathbb{Z})$ but $H^k(X) = 0$ for $k \neq 0$.
- Put $A = \mathrm{Map}(X, \mathbb{Z})$. As X is discrete, any continuous map $u\colon \Delta_k \to X$ is constant, so $S_k(X) \simeq X$ and $C^k(X) = \mathrm{Map}(S_k(X), \mathbb{Z}) \simeq A$.
- If $u\colon \Delta_{k+1} \to X$ is constant with value x, then $u \circ \delta_i\colon \Delta_k \to X$ is also constant, with the same value.
- The formula for $d\colon C^k(X) = A \to A = C^{k+1}(X)$ just becomes

$$d(f)(x) = \sum_{i=0}^{k+1} (-1)^i f(x).$$

- If k is even: all terms cancel out in pairs, giving $d(f)(x) = 0$.
  If k is odd: there is one term left over, giving $d(f)(x) = f(x)$.
- Thus, the full sequence of groups $C^k(X)$ and homomorphisms d looks like

$$\cdots \to 0 \to 0 \to C^0(X) = A \xrightarrow{0} A \xrightarrow{1} A \xrightarrow{0} A \xrightarrow{1} A \to \cdots$$

- For $k < 0$ we have $Z^k = B^k = C^k(X) = 0$ so $H^k(0) = 0$.
- $Z^0 = A$ but $B^0 = 0$ so $H^0(X) = Z^0/B^0 = A/0 = A = \mathrm{Map}(X, \mathbb{Z})$.
- For $k > 0$, if k is even we have $Z^k = B^k = A$ and if k is odd we have $Z^k = B^k = 0$. In both cases we have $Z^k = B^k$ so $H^k(X) = Z^k/B^k = 0$.

The cup product

- Given $f \in C^n(X)$ and $g \in C^m(X)$ we need to define $fg \in C^{n+m}(X)$.
- Here $C^{n+m}(X) = \mathrm{Map}(S_{n+m}(X), \mathbb{Z})$ and $S_{n+m}(X)$ is the set of continuous maps $w\colon \Delta^{n+m} \to X$, so for each such w we must define $(fg)(w) \in \mathbb{Z}$.
- Define $\Delta^n \xrightarrow{\lambda} \Delta^{n+m} \xleftarrow{\rho} \Delta^m$ by

$$\lambda(x_0, \dots, x_n) = (x_0, \dots, x_n, 0, \dots, 0) \qquad \rho(y_0, \dots, y_m) = (0, \dots, 0, y_0, \dots, y_m).$$

- Now $w\lambda\colon \Delta_n \to X$ so $w\lambda \in S_n(X)$ so $f(w\lambda) \in \mathbb{Z}$.
- Also $w\rho\colon \Delta_m \to X$ so $w\rho \in S_m(X)$ so $g(w\rho) \in \mathbb{Z}$.
- We define $(fg)(w) = f(w\lambda)\, g(w\rho) \in \mathbb{Z}$.
- We also define $1 \in C^0(X) = \mathrm{Map}(X, \mathbb{Z})$ to be constant with value 1.
- These definitions make $C^*(X)$ into a differential graded ring: multiplication is distributive and associative with 1 as a two-sided unit, and $d(1) = 0$, and $d(fg) = d(f)g + (-1)^n f\, d(g)$.
- The proof is an exercise.
- As discussed previously, there is an induced ring structure on $H^*(X)$.
- $H^*(X)$ is graded-commutative even though $C^*(X)$ is not.
  The proof is harder, to be discussed later.

Functoriality

- A *cochain map* between cochain complexes U^* and V^* is a system of homomorphisms $\phi\colon U^n \to V^n$ with $d\phi = \phi d\colon U^n \to V^{n+1}$.
- For such ϕ, we see that $\phi(Z^n(U^*)) \leq Z^n(V^*)$ and $\phi(B^n(U^*)) \leq B^n(V^*)$ so we have an induced homomorphism $H^n(\phi)\colon H^n(U^*) \to H^n(V^*)$.
- This is functorial: $H^n(1) = 1$ and $H^n(\psi\phi) = H^n(\psi)H^n(\phi)$ for cochain maps $U^* \xrightarrow{\phi} V^* \xrightarrow{\psi} W^*$.
- If U^* and V^* are DGRs: a DGR morphism is a cochain map that also preserves products. For such ϕ, the induced map $H^*(\phi)\colon H^*(U^*) \to H^*(V^*)$ is a graded ring homomorphism.
- Now let $p\colon X \to Y$ be a continuous map. For $f \in C^n(Y)$ and $u \in S_n(X) = \mathrm{Cont}(\Delta^n, X)$ we have $pu \in \mathrm{Cont}(\Delta^n, Y) = S_n(Y)$ and so $f(pu) \in \mathbb{Z}$. We define $p^*(f) \in C^n(X)$ by $p^*(f)(u) = f(pu)$.
- Using $p \circ (u \circ \delta_i) = (p \circ u) \circ \delta_i$, we see that $p^*(d(f)) = d(p^*(f))$ in $C^{n+1}(X)$. Thus, p^* is a cochain map.
- Using $p \circ (w \circ \lambda) = (p \circ w) \circ \lambda$ and $p \circ (w \circ \rho) = (p \circ w) \circ \rho$, we see that $p^*(fg) = p^*(f)p^*(g)$ in $C^{n+m}(X)$. Thus, p^* is a morphism of DGRs, and so induces a graded ring homomorphism $H^*(Y) \to H^*(X)$, which we also call p^*.

Topological homotopy

- A *homotopy* between continuous maps $f_0, f_1\colon X \to Y$ is a continuous map $F\colon [0,1] \times X \to Y$ with $F(0,x) = f_0(x)$ and $F(1,x) = f_1(x)$ for all $x \in X$.
- We say that f_0 and f_1 are *homotopic* if such a homotopy exists.
- Exercise: this is an equivalence relation (written $f_0 \simeq f_1$).
  Key point: given homotopies $F_0\colon f_0 \simeq f_1$ and $F_1\colon f_1 \simeq f_2$ we can put

$$F(t,x) = \begin{cases} F_0(2t,x) & \text{if } 0 \leq t \leq \frac{1}{2} \\ F_1(2t-1,x) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

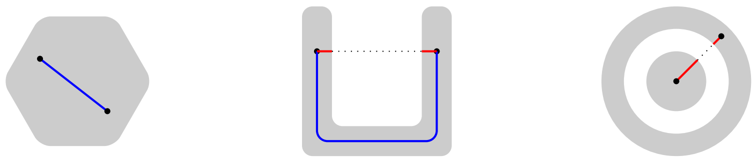
- Example: we can define $f\colon S^n \to S^n$ by $f(x) = -x$.
  If $n = 2m-1$ then $S^n = \{z \in \mathbb{C}^m \mid \|z\| = 1\}$ and we can define $F\colon 1_{S^n} \simeq f$ by $F(t,z) = e^{\pi i t}z$.
  If n is even then cohomology shows that $1_{S^n} \not\simeq f$.
- We can define $p_n\colon S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \to S^1$ by $p_n(z) = z^n$.
  Fact: any $f\colon S^1 \to S^1$ is homotopic to p_n for a unique n.
- $F(t,x) = (1-t)f_0(x) + t\,f_1(x)$ gives a linear homotopy $f_0 \simeq f_1$ **only if** $Y \subseteq \mathbb{R}^N$ and the line segment from $f_0(x)$ to $f_1(x)$ is always contained in Y.
- Say $Y \subseteq \mathbb{R}^N$ is *convex* if $Y \neq \emptyset$ and every segment with endpoints in Y is contained in Y. If so, all maps $X \to Y$ are homotopic.

Topological homotopy

- Homotopy is compatible with composition. In detail, if $X \xrightarrow{f_0, f_1} Y \xrightarrow{g_0, g_1} Z$ and we have homotopies $F\colon f_0 \simeq f_1$ and $G\colon g_0 \simeq g_1$, then we can define $K\colon g_0 f_0 \simeq g_1 f_1$ by $K(t,x) = G(t, F(t,x))$.
- We write $[X,Y] = \mathrm{Cont}(X,Y)/\simeq$ for the set of homotopy classes.
- Example: for $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, every $f\colon S^1 \to S^1$ is homotopic to $p_n(z) = z^n$ for a unique $n \in \mathbb{Z}$, so $[S^1, S^1] \simeq \mathbb{Z}$.
- There is a well-defined composition $[Y,Z] \times [X,Y] \to [X,Z]$ and thus a category hTop of spaces and homotopy classes of maps.
- Maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ are *homotopy inverse* if $gf \simeq 1_X$ and $fg \simeq 1_Y$, i.e. $[g]$ is inverse to $[f]$ in hTop.
- Say $f\colon X \to Y$ is a *homotopy equivalence* if it has a homotopy inverse, i.e. it becomes an isomorphism in hTop.
- Say X and Y are *homotopy equivalent* if there is a homotopy equivalence $f\colon X \to Y$, i.e. $X \simeq Y$ in hTop.
- Example: define $S^{n-1} \xrightarrow{i} \mathbb{R}^n \setminus \{0\} \xrightarrow{p} S^{n-1}$ by $i(x) = x$ and $r(y) = y/\|y\|$.
  Define $F\colon [0,1] \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}^n \setminus \{0\}$ by $F(t,y) = \|y\|^{-t}y$.
  Then $pi = 1$ and $F\colon 1 \simeq ip$ so i and p are mutually inverse homotopy equivalences, and S^{n-1} and $\mathbb{R}^n \setminus \{0\}$ are homotopy equivalent spaces.

Contractible spaces

- Say X is *contractible* iff it is homotopy equivalent to $1 = \{0\}$.
- Exercise: X is contractible iff $X \neq \emptyset$ and $1\colon X \to X$ is homotopic to a constant map.
- Exercise: any contractible space is path-connected.
- Example: any convex subset of $\mathbb{R}^N$ is contractible, and any space homeomorphic to a contractible space is contractible.



convex | contractible, but not convex | not contractible

- The following spaces are convex and so contractible: $\mathbb{R}^n$, B^n, Δ^n, $[0,1]^n$.
- Slogan: in homotopy theory, a contractible space of choices is as good as a unique choice.

Cochain homotopy

- Let $\phi, \phi' \colon U^* \to V^*$ be cochain maps. A *chain homotopy* from ϕ to ϕ' is a system of homomorphisms $\sigma \colon U^n \to V^{n-1}$ with $d\sigma + \sigma d = \phi' - \phi$. We say that ϕ and ϕ' are *chain homotopic* if such a chain homotopy exists.
- Exercise: this is an equivalence relation (written $\phi \simeq \phi'$).
- Exercise: this relation is compatible with composition:
  If $U^* \xrightarrow{\phi,\phi'} V^* \xrightarrow{\psi,\psi'} W^*$ and $\sigma \colon \phi \simeq \phi'$ and $\tau \colon \psi \simeq \psi'$ then $\psi\sigma + \tau\phi' \colon \psi\phi \simeq \psi'\phi'$.
- Claim: if $\sigma \colon \phi \simeq \phi'$ then $H^*(\phi) = H^*(\phi') \colon H^*(U^*) \to H^*(V^*)$.
- Proof: consider an element $z \in Z^n(U^*)$ (so $d(z) = 0$). Then $H^n(\phi')([z]) - H^n(\phi)([z]) = [\phi'(z)] - [\phi(z)] = [(\phi' - \phi)(z)] = [d(\sigma(z)) + \sigma(d(z))] = [d(\cdot) + 0] = 0$.
- Proposition: a topological homotopy $F \colon [0,1] \times X \to Y$ from f_0 to f_1 gives a chain homotopy between $f_0^*, f_1^* \colon C^*(Y) \to C^*(X)$, so $f_0^* = f_1^* \colon H^*(Y) \to H^*(X)$.
- Core of proof: divide $[0,1] \times \Delta^n$ into copies of Δ^{n+1}, and think about the boundary of this space.
- Corollary: if X is homotopy equivalent to Y, then $H^*(X) \simeq H^*(Y)$.
- Example: if X is contractible then $H^n(X) = 0$ except $H^0(X) = \mathbb{Z}$.

The Mayer-Vietoris context

- Consider a space X with open subsets $U, V \subseteq X$.
- How are $H^*(U)$, $H^*(V)$, $H^*(U \cup V)$ and $H^*(U \cap V)$ related?

$$
\begin{tikzcd}
U \cap V \arrow[r, "i"] \arrow[d, "j"'] & U \arrow[d, "k"] \\
V \arrow[r, "l"'] & U \cup V
\end{tikzcd}
\qquad
\begin{tikzcd}
H^*(U \cap V) & H^*(U) \arrow[l, "i^*"'] \\
H^*(V) \arrow[u, "j^*"] & H^*(U \cup V) \arrow[l, "l^*"] \arrow[u, "k^*"']
\end{tikzcd}
$$

$$H^{n-1}(U \cap V) \xrightarrow{\delta} H^n(U \cup V) \xrightarrow{\left[\begin{smallmatrix} k^* \\ l^* \end{smallmatrix}\right]} H^n(U) \times H^n(V) \xrightarrow{[\,i^* \;\; -j^*\,]} H^n(U \cap V) \xrightarrow{\delta} H^{n+1}(U \cup V)$$

There is a non-obvious map δ extending the diagram as shown,
and this makes the sequence exact,
i.e. the image of each map is the kernel of the next.

Also: we have a ring map $\alpha = (ki)^* = (lj)^* \colon H^*(U \cup V) \to H^*(U \cap V)$, and $\delta(\alpha(a)b) = (-1)^n a\, \delta(b)$ for $a \in H^n(U \cup V)$ and $b \in H^m(U \cap V)$.

Exact sequences

- A sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is *exact* if $\mathrm{img}(\alpha) = \ker(\beta)$ (implies $\beta\alpha = 0$)
- The sequence is *short exact* if also α is injective and β is surjective.
- $A \xrightarrow{\alpha} B \xrightarrow{0} C$ is exact iff α is surjective; so $A \xrightarrow{\alpha} B \to 0$ is exact iff α is surjective.
- $A \xrightarrow{0} B \xrightarrow{\beta} C$ is exact iff β is injective; so $0 \to B \xrightarrow{\beta} C$ is exact iff β is injective.
- $A \xrightarrow{0} B \xrightarrow{\beta} C \xrightarrow{0} D$ is exact iff β is an isomorphism; so $0 \to B \xrightarrow{\beta} C \to 0$ is exact iff β is an isomorphism.
- $A \xrightarrow{0} B \xrightarrow{0} C$ is exact iff $B = 0$; so $0 \to B \to 0$ is exact iff $B = 0$.
- A cochain complex $U^* = (\cdots \to U^{-2} \to U^{-1} \to U^0 \to U^1 \to U^2 \to \cdots)$ is exact iff $H^*(U^*) = 0$.
- *Split* short exact sequence:
  $A \xrightarrow{i} A \oplus B \xrightarrow{p} B$ with $i(a) = (a, 0)$ and $p(a, b) = b$.
- There is a short exact sequence $\mathbb{Z}/n \xrightarrow{i} \mathbb{Z}/nm \xrightarrow{p} \mathbb{Z}/m$ with $i(a \pmod{n}) = am \pmod{nm}$ and $p(a \pmod{nm}) = a \pmod{m}$. This is split iff n and m are coprime.
- For $N \leq M$, $\;N \xrightarrow{\text{inc}} M \xrightarrow{\text{proj}} M/N\;$ is short exact.
- If $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is short exact then $A \simeq \alpha(A) \leq B$ and $B/\alpha(A) \simeq C$ so $|B| = |A||B|$.

The Snake Lemma

- Let $U^* \xrightarrow{i} V^* \xrightarrow{p} W^*$ be a SES of cochain complexes and chain maps (so $d^2 = 0$; $di = id$ and $dp = pd$; $\mathrm{img}(i) = \ker(p)$; i injective, p surjective)
- Claim: there are maps $\delta \colon H^n(W^*) \to H^{n+1}(U^*)$ giving an exact sequence

$$\cdots \to H^{n-1}(W^*) \xrightarrow{\delta} H^n(U^*) \xrightarrow{i_*} H^n(V^*) \xrightarrow{p_*} H^n(W^*) \xrightarrow{\delta} H^{n+1}(U^*) \to \cdots$$

- Idea: $\delta = i^{-1}dp^{-1} = \;(H^n(W^*) \overset{p^{-1}}{\dashrightarrow} V^n \xrightarrow{d} V^{n+1} \overset{i^{-1}}{\dashrightarrow} H^{n+1}(U^*))$
- Definition: a *snake* is (c, w, v, u, a) where
  (1) $c \in H^n(W^*)$; (2) $w \in Z^n(W^*)$ with $c = [w]$;
  (3) $v \in V^n$ with $p(v) = w$; (4) $u \in Z^{n+1}(U^*)$ with $i(u) = d(v)$;
  (5) $a = [u] \in H^{n+1}(U^*)$.
- Idea: v is a choice of $p^{-1}(c)$, a is a choice of $i^{-1}(d(v)) = i^{-1}(d(p^{-1}(c)))$.
- Claim: for $c \in H^n(W^*)$, there is a snake (c, w, v, u, a) starting with c. Any two choices have the same a so we can define $\delta(c) = a$ giving $\delta \colon H^n(W^*) \to H^{n+1}(U^*)$.
- Proof: By definition of $H^n(W^*)$ there exists w as in (2). As p is surjective there exists v as in (3). Now $p(d(v)) = d(p(v)) = d(w) = 0$ so $d(v) \in \ker(p) = \mathrm{img}(i)$ so there exists $u \in U^{n+1}$ with $i(u) = d(v)$. Also $i(d(u)) = d(i(u)) = d^2(v) = 0$ but i is injective so $d(u) = 0$ so u is as in (4). We define $a = [u]$ so (5) holds. Uniqueness is similar.

The Snake Lemma

- Let $U^* \xrightarrow{i} V^* \xrightarrow{p} W^*$ be a SES of cochain complexes and chain maps
- $\delta\colon H^n(W^*) \to H^{n+1}(U^*)$ with $\delta(c) = a$ iff there is a snake (c, w, v, u, a)
  i.e. (1) $c \in H^n(W^*)$; (2) $w \in Z^n(W^*)$ with $c = [w]$;
  (3) $v \in V^n$ with $p(v) = w$; (4) $u \in Z^{n+1}(U^*)$ with $i(u) = d(v)$;
  (5) $a = [u] \in H^{n+1}(U^*)$.
- Claim: the following sequence is exact:

  $$\cdots \to H^{n-1}(W^*) \xrightarrow{\delta} H^n(U^*) \xrightarrow{i_*} H^n(V^*) \xrightarrow{p_*} H^n(W^*) \xrightarrow{\delta} H^{n+1}(U^*) \to \cdots$$

  i.e. $i_*\delta = 0$, $p_*i_* = 0$, $\delta p_* = 0$,
  $\ker(i_*) \leq \mathrm{img}(\delta)$, $\ker(p_*) \leq \mathrm{img}(i_*)$, $\ker(\delta) \leq \mathrm{img}(p_*)$.
- For $i_*\delta = 0$: $i_*(\delta(c)) = i_*([u]) = [i(u)] = [d(v)] = 0$.
- For $p_*i_* = 0$: $p_*(i_*([u])) = p_*([i(u)]) = [p(i(0))] = [0] = 0$.
- For $\delta p_* = 0$: if $v \in Z^n(V^*)$ then $d(v) = 0 = i(0)$ so we have a snake $(p_*([v]), p(v), v, 0, 0)$.
- For $\ker(i_*) \leq \mathrm{img}(\delta)$: suppose $u \in Z^n(U^*)$ with $i_*([u]) = 0$. Then $[i(u)] = 0$ so $i(u) = d(v)$ for some $v \in V^{n-1}$. Then $d(p(v)) = p(d(v)) = p(i(u)) = 0$ so we have a snake $([p(v)], p(v), v, u, [u])$ giving $[u] = \delta([p(v)]) \in \mathrm{img}(\delta)$.
- The rest is similar.

The Mayer-Vietoris Sequence

- Suppose $U \xrightarrow{i} X \xleftarrow{j} V$ are inclusions of open sets with $X = U \cup V$.

$$
\begin{aligned}
S_n^0(X) &= \{u\colon \Delta^n \to X \mid u(\Delta^n) \subseteq U \cap V\}\\
S_n^1(X) &= \{u\colon \Delta^n \to X \mid u(\Delta^n) \subseteq U, u(\Delta^n) \not\subseteq V\}\\
S_n^2(X) &= \{u\colon \Delta^n \to X \mid u(\Delta^n) \not\subseteq U, u(\Delta^n) \subseteq V\}\\
S_n^3(X) &= \{u\colon \Delta^n \to X \mid u(\Delta^n) \not\subseteq U, u(\Delta^n) \not\subseteq V\} = \{\text{large } n\text{-simplices}\}\\
A_n^k &= \mathrm{Map}(S_n^k, \mathbb{Z})\\
C^*(X) &= A_0^* \times A_1^* \times A_2^* \times A_3^* =: A^*\\
C^*(U) &= A_0^* \times A_1^* = A^*/I^* \text{ where } I^* = A_2^* \times A_3^*\\
C^*(V) &= A_0^* \times A_2^* = A^*/J^* \text{ where } J^* = A_1^* \times A_3^*\\
C^*(U \cap V) &= A_0^* = A^*/(I^* + J^*)\\
C^*_{\mathrm{small}}(X) &= A_0^* \times A_1^* \times A_2^* = A^*/(I^* \cap J^*)
\end{aligned}
$$

- We have a short exact sequence

  $$C^*_{\mathrm{small}}(X) \xrightarrow{\left[\begin{smallmatrix} k^* \\ l^* \end{smallmatrix}\right]} C^*(U) \times C^*(V) \xrightarrow{[i^* \ -j^*]} C^*(U \cap V)$$

  giving a Mayer-Vietoris type sequence
  $\cdots \to H^{n-1}(U \cap V) \to H^n_{\mathrm{small}}(X) \to H^n(U) \times H^n(V) \to H^n(U \cap V) \to H^{n+1}_{\mathrm{small}}(X) \to \cdots$

The Mayer-Vietoris Sequence

- Suppose $U \xrightarrow{i} X \xleftarrow{j} V$ are inclusions of open sets with $X = U \cup V$.
- Put $A^* = C^*(X)$ and $C^*_{\mathrm{small}}(X) = A^*/K^*$ where $K^* = I^* \cap J^* = \ker(i^*) \cap \ker(j^*) = C^*_{\mathrm{big}}(X)$.
- The short exact sequence $K^* \to A^* \to A^*/K^*$ gives an exact sequence

  $$H^n(K^*) \to H^n(A^*) = H^n(X) \to H^n(A^*/K^*) = H^n_{\mathrm{small}}(X) \xrightarrow{\delta} H^{n+1}(K^*)$$

- Claim: $H^*(K^*) = 0$. Given this, the above gives $H^*(X) = H^*_{\mathrm{small}}(X)$ so we have the Mayer-Vietoris sequence as originally stated.
- Why is $H^*(K^*) = 0$? First $K^0 = 0$ so $H^0(K^*) = 0$ and $H^1(K^*) = Z^1(K^*)$.
- Consider a path $u\colon [0,1] = \Delta^1 \to X$ and let u_0, u_1 be the first and second halves. Define $p\colon \Delta^2 \to \Delta^1$ by $p(t_0, t_1, t_2) = (t_0 + t_1/2, t_1/2 + t_2)$.
  If $f \in Z^1(K^*)$ then using $(df)(u \circ p) = 0$ we get $f(u) = f(u_0) + f(u_1)$.
  Repeat: $f(u)$ is a sum of 2^N terms, each f applied to a small piece of u.
  Eventually all the pieces lie in U or in V, so $f(u) = 0$.
- To prove $H^n(K^*) = 0$ in general, we need to subdivide Δ^n into smaller copies of Δ^n and also define a map $\Delta^{n+1} \to \Delta^n$ analogous to p.
  This can be done by explicit combinatorics or by a more abstract method ("acyclic models").

Cohomology of spheres

- Claim: For $n \geq 0$ there is an element $u_n \in H^n(S^n)$ such that $H^*(S^n) = \mathbb{Z} \oplus \mathbb{Z}u_n$.
- For $n = 0$: the space $S^0 = \{1, -1\}$ is discrete, so $H^n(S^0) = 0$ for $n \neq 0$ and $H^0(S^0) = \mathrm{Map}(S^0, \mathbb{Z})$. We put $u_0(1) = 01$ and $u_0(-1) = 1$ so $H^0(S^0) = \mathbb{Z} \oplus \mathbb{Z}u_0$.
- For $n > 0$, we put $N = (0, \dots, 0, 1) \in S^n$ and $U = S^n \setminus \{-N\}$ and $V = S^n \setminus \{N\}$ so $S^n = U \cup V$.
- For $(x, t) \in U \cap V = S^n \setminus \{N, -N\}$ we have $\|x\|^2 + t^2 = 1$ with $|t| < 1$ so $x \neq 0$; so we can define $r\colon U \cap V \to S^{n-1}$ by $r(x, t) = x/\|x\|$.
- We also have $\delta\colon H^{n-1}(U \cap V) \to H^n(S^n)$ and we put $u_n = \delta(r^*(u_{n-1}))$.
- Stereographic projection: $U \simeq V \simeq \mathbb{R}^n$ (contractible) so $H^0(U) = H^0(V) = \mathbb{Z}$ but $H^n(U) = H^n(V) = 0$ otherwise.
- $i\colon S^{n-1} \to U \cap V$ by $i(x) = (x, 0)$ has $ri = 1$ and $h\colon ir \simeq 1$ by $h(s, (x, t)) = (x, st)/\|(x, st)\|$. Thus $H^*(U \cap V) \simeq H^*(S^{n-1}) = \mathbb{Z} \oplus \mathbb{Z}u_{n-1}$.

Cohomology of the circle

$$\begin{tikzcd}
0 \arrow[r] & H^0(S^1) \arrow[r, "\alpha"] & H^0(U)\times H^0(V) \arrow[r, "\beta"] & H^0(S^0) \arrow[dll, "\delta"] \\
& H^1(S^1) \arrow[r] & H^1(U)\times H^1(V) \arrow[r] & H^1(S^0) \arrow[dll, "\delta"] \\
& H^2(S^1) \arrow[r] & H^2(U)\times H^2(V) \arrow[r] & H^2(S^0) \arrow[dll, "\delta"] \\
& H^3(S^1) \arrow[r] & H^3(U)\times H^3(V) \arrow[r] & H^3(S^0) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\alpha"] & \mathbb{Z}\times\mathbb{Z} \arrow[r, "\beta"] & \mathbb{Z}\{1, u_0\} \arrow[dll, "\delta"] \\
& H^1(S^1) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "\delta"] \\
& H^2(S^1) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "\delta"] \\
& H^3(S^1) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

- $\alpha(n) = (n, n)$ and $\beta(p, q) = (q - p).1$.
- It follows that $H^0(S^1) = \mathbb{Z}$ and $H^1(S^1) = \mathbb{Z}u_1$ and $H^n(S^1) = 0$ otherwise, as claimed.

Cohomology of the sphere

$$\begin{tikzcd}
0 \arrow[r] & H^0(S^2) \arrow[r, "\alpha"] & H^0(U)\times H^0(V) \arrow[r, "\beta"] & H^0(S^1) \arrow[dll, "\delta"] \\
& H^1(S^2) \arrow[r] & H^1(U)\times H^1(V) \arrow[r] & H^1(S^1) \arrow[dll, "\delta"] \\
& H^2(S^2) \arrow[r] & H^2(U)\times H^2(V) \arrow[r] & H^2(S^1) \arrow[dll, "\delta"] \\
& H^3(S^2) \arrow[r] & H^3(U)\times H^3(V) \arrow[r] & H^3(S^1) \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\alpha"] & \mathbb{Z}\times\mathbb{Z} \arrow[r, two heads, "\beta"] & \mathbb{Z} \arrow[dll, "\delta=0"] \\
& H^1(S^2) \arrow[r] & 0 \arrow[r] & \mathbb{Z}u_1 \arrow[dll, "\delta"] \\
& H^2(S^2) \arrow[r] & 0 \arrow[r] & 0 \arrow[dll, "\delta"] \\
& H^3(S^2) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

- $\alpha(n) = (n, n)$ and $\beta(p, q) = (q - p).1$.
- It follows that $H^0(S^2) = \mathbb{Z}$ and $H^2(S^2) = \mathbb{Z}u_2$ and $H^n(S^2) = 0$ otherwise, as claimed.

Distinguishing spheres and euclidean spaces

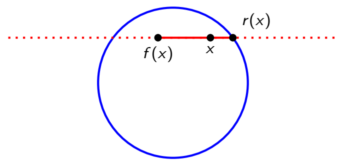
- We proved: $H^*(S^n) = \mathbb{Z} \oplus \mathbb{Z}u_n$ with $u_n \in H^n(S^n)$.
- Thus: if $n \neq m$ then $H^*(S^n) \not\simeq H^*(S^m)$ as graded rings so S^n is not homotopy equivalent to S^m.
- Recall that $\mathbb{R}^{n+1} \setminus \{0\}$ is homotopy equivalent to S^n. Thus, if $n \neq m$ then $\mathbb{R}^{n+1} \setminus \{0\}$ is not homotopy equivalent to $\mathbb{R}^{m+1} \setminus \{0\}$.
- Also $\mathbb{R}^0 \setminus \{0\} = \emptyset$, so if $p \neq q$ then $\mathbb{R}^p \setminus \{0\}$ is not homotopy equivalent to $\mathbb{R}^q \setminus \{0\}$.
- Given a homeomorphism $f : \mathbb{R}^p \to \mathbb{R}^q$, we can define $g(x) = f(x) - f(0)$ with $g^{-1}(y) = f^{-1}(y + f(0))$; this gives another homeomorphism with $g(0) = 0$. This in turn gives a homeomorphism $\mathbb{R}^p \setminus \{0\} \to \mathbb{R}^q \setminus \{0\}$ so $p = q$.
- Conclusion: if $p \neq q$ then $\mathbb{R}^p$ is not homeomorphic to $\mathbb{R}^q$.
- This is very easy to believe but very hard to prove without cohomology.

The Brouwer fixed point theorem

- **Lemma:** if $i : S^{n-1} \to B^n$ is the inclusion then there is no continuous map $r : B^n \to S^{n-1}$ with $ri = 1 : S^{n-1} \to S^{n-1}$.
- **Proof:** Cases $n = 0, 1$ (with $S^{-1} = \emptyset$) are easy so take $n > 1$.
- If $ri = 1$ then the composite

$$\mathbb{Z} = H^{n-1}(S^{n-1}) \xrightarrow{i^*} H^{n-1}(B^n) = 0 \xrightarrow{r^*} H^{n-1}(S^{n-1} = \mathbb{Z}$$

  is the identity, but that is impossible. □
- **Theorem (Brouwer):** if $f : B^n \to B^n$ is continuous, then there exists $x \in B^n$ with $f(x) = x$.
- **Proof:** suppose not. Then for each x we can draw a line from $f(x)$ to x and extend it until we hit the boundary at a point $r(x) \in S^{n-1}$.



- If $x \in S^{n-1}$ we just have $r(x) = x$. One can check that r is continuous, so this contradicts the lemma. □

Cohomology of $X \times Y$

Suppose we have two spaces X and Y, and thus projections $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$. Given $a \in H^r(X)$ and $b \in H^s(Y)$ we define $a \times b = p^*(a)q^*(b) \in H^{r+s}(X \times Y)$; this is called the *external product* of a and b.

This construction gives a map $\mu\colon H^*(X) \otimes H^*(Y) \to H^*(X \times Y)$, with $\mu(a \otimes b) = a \times b$.

Here $M \otimes N \simeq N \otimes M$; $\quad (L \oplus M) \otimes N \simeq (L \otimes N) \oplus (M \otimes N)$; $\quad \mathbb{Z} \otimes M \simeq M$; $(\mathbb{Z}/r) \otimes M = M/rM$; $\quad \mathbb{Z}^n \otimes \mathbb{Z}^m \simeq \mathbb{Z}^{nm}$; $\quad \mathbb{Z}/r \otimes \mathbb{Z}/s \simeq \mathbb{Z}/\gcd(r,s)$

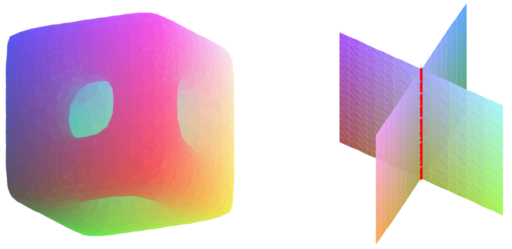
The map μ is an isomorphism if each group $H^r(X)$ is free and finitely generated (this is a special case of the *Künneth theorem*).

Now consider the inclusions $X \xrightarrow{i} X \amalg Y \xleftarrow{j} Y$ and the resulting map $H^*(X \amalg Y) \to H^*(X) \times H^*(Y)$, given by $a \mapsto (i^*(a), j^*(a))$. This is easily seen to be an isomorphism.

The definition of a manifold

Definition

A *topological manifold of dimension n* is a second countable, Hausdorff topological space M such that each point $x \in M$ has an open neighbourhood $U \subseteq M$ such that U is homeomorphic to $\mathbb{R}^n$.



The space on the left is a manifold of dimension 2; the one on the right is not.

Open subsets of $\mathbb{R}^n$

Example

Let U be the open ball of radius $\epsilon > 0$ around a point $x \in \mathbb{R}^n$. Then there is a homeomorphism $f\colon U \to \mathbb{R}^n$:

$$f(y) = \frac{y - x}{1 - \|y - x\|^2/\epsilon^2} \qquad\qquad f^{-1}(z) = x + \frac{\sqrt{\epsilon^2 + 4\|z\|^2} - \epsilon}{2\|z\|^2}\epsilon z$$

It follows that any open subspace of $\mathbb{R}^n$ is an n-dimensional topological manifold.
An interesting special case is

$$F_n\mathbb{C} := \{z \in \mathbb{C}^n \mid z_i \neq z_j \text{ when } i \neq j\}.$$

This can be viewed as an open subspace of $\mathbb{C}^n \simeq \mathbb{R}^{2n}$; we will study its cohomology later.

Vector spaces

Convention

Many examples below will involve vector spaces. Everywhere in these notes, vector spaces are assumed finite dimensional unless otherwise specified, and the scalar field is $\mathbb{R}$ unless otherwise specified.

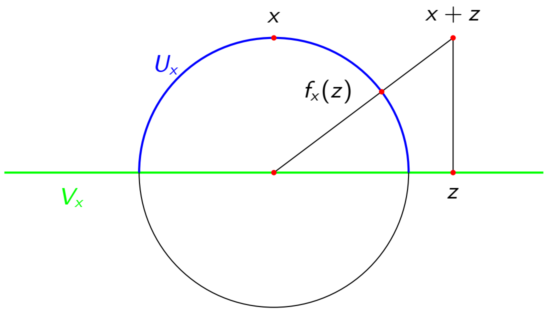
Example

Let V be a vector space of dimension n. There is a natural topology on V (the smallest one for which all linear maps $V \to \mathbb{R}$ are continuous) and with this topology V is homeomorphic to $\mathbb{R}^n$. Thus V is a topological manifold.

Spheres

Example

Now suppose that V is equipped with an inner product, and define the sphere $S(V)$ as $\{x \in V \mid \|x\| = 1\}$.
For $x \in S(V)$ put $U_x = \{y \in S(V) \mid \langle x, y\rangle > 0\}$ and $V_x = \{z \mid \langle x, z\rangle = 0\}$.
Define $f_x \colon V_x \to U_x$ by $f_x(z) = (x + z)/\sqrt{1 + z^2}$.

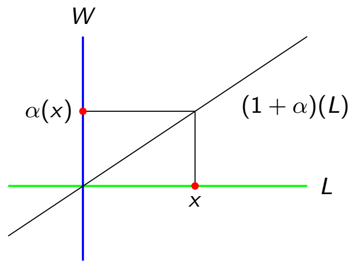


One can check that this is a homeomorphism, and also V_x is a vector space of dimension $n-1$ so it is homeomorphic to $\mathbb{R}^{n-1}$. It follows that $S(V)$ is a manifold of dimension $n-1$. It is easy to see that it is compact.

Complex projective space

Let V have dimension m over $\mathbb{C}$. Put $PV = \{\text{lines in } V\}$.
Define $q \colon V^\times = V \setminus \{0\} \to PV$ by $q(x) = [x] = \mathbb{C}x$. This is surjective, and we give PV the quotient topology. Claim: this makes PV a topological manifold.
Indeed, given a line $L \in PV$ choose W with $V = L \oplus W$, and put $U = \{M \in PV \mid M \cap W = 0\}$. Then U is an open neighbourhood of L in PV.
We can define $f = f_{L,W} \colon \mathrm{Hom}(L, W) \to PV$ by

$$f(\alpha) = \mathrm{graph}(\alpha \colon L \to W) = (1 + \alpha)(L) \leq L + W = V.$$



One can check that this gives a homeomorphism from $\mathrm{Hom}(L, W) \simeq \mathbb{C}^{n-1} \simeq \mathbb{R}^{2n-2}$ to U, so U is a chart domain around L.
For $V = \mathbb{C}^{m+1}$: $PV = \mathbb{C}P^m$, $[z_0 : \cdots : z_m] = \mathbb{C}.(z_0, \ldots, z_m)$

Some projective varieties

Suppose that $m \leq n$. The *Milnor hypersurface* in $\mathbb{C}P^m \times \mathbb{C}P^n$ is the space

$$H_{m,n} = \{([z], [w]) \in \mathbb{C}P^m \times \mathbb{C}P^n \mid \sum_{i=0}^{m} z_i w_i = 0\}.$$

Suppose that $d > 2$. The *Fermat hypersurface of degree d* in $\mathbb{C}P^m$ is

$$X_{d,m} = \{[z] \in \mathbb{C}P^m \mid \sum_{i=0}^{m} z_i^d = 0\}.$$

Consider the space

$$C = \{[x : y : z] \in \mathbb{C}P^2 \mid y^2 z = x(x - z)(x + z)\}.$$

This is an example of an *elliptic curve*. It is homeomorphic to the torus $S^1 \times S^1$.

Grassmannians and flag varieties

Let $G_k(V)$ be the set of k-dimensional subspaces of $V \simeq \mathbb{C}^d$.
(So $PV = G_1(V)$.)
This is again a compact manifold, of dimension $2k(d-k)$.

Indeed, given $A \in G_k(V)$ we can choose a subspace $B \in G_{d-k}(V)$ with $V = A \oplus B$. We find that the set $U = \{A' \in G_k(V) \mid A' \cap B = 0\}$ is an open neighbourhood of A, and that we have a homeomorphism $\mathrm{Hom}(A, B) \to U$ given by $\alpha \mapsto \mathrm{graph}(\alpha) = (1 + \alpha)(A)$.

A *complete flag* in V is a sequence of complex subspaces $0 = W_0 < W_1 < \ldots < W_d = V$ such that $\dim(W_k) = k$ for all k. The space of complete flags is written $\mathrm{Flag}(V)$; it is again a compact manifold, of dimension $d^2 - d$.

A flag W in $V = \mathbb{C}^d$ is *bounded* if $W_k \leq \mathbb{C}^{k+1}$ for all k. The set B_d of bounded flags is a manifold of dimension $2d - 2$. It is an example of a *toric variety*: there is an action of the group $(\mathbb{C}^\times)^{d-1}$ that is nearly free and nearly transitive.

Unitary groups

Let V be a complex vector space of dimension d.
Suppose we have a Hermitian inner product (so that $\langle u,v\rangle = \overline{\langle v,u\rangle}$ and $z\langle u,v\rangle = \langle zu,v\rangle = \langle u,\overline{z}v\rangle$ when $z\in\mathbb{C}$ and $u,v\in V$).
Any endomorphism α of V has an adjoint $\alpha^\dagger$, with $\langle\alpha(u),v\rangle = \langle u,\alpha^\dagger(v)\rangle$. Put

$$U(V) = \{\alpha\in\mathrm{Aut}(V) \mid \alpha^\dagger = \alpha^{-1}\} = \text{ the unitary group of } V.$$
$$\mathfrak{u}(V) = \{\beta\in\mathrm{End}(V) \mid \beta^\dagger = -\beta\}.$$

After choosing an orthonormal basis for V, it is not hard to check that $\mathfrak{u}(V)$ is a real vector space of dimension d^2.
Also, if $\beta\in\mathfrak{u}(V)$ we see that the eigenvalues of β are purely imaginary, so that the maps $1\pm\beta/2$ are invertible. For any $\alpha\in U(V)$ we define $f_\alpha\colon \mathfrak{u}(V)\to\mathrm{Aut}(V)$ by

$$f_\alpha(\beta) = (1+\beta/2)(1-\beta/2)^{-1}\alpha.$$

One checks that this gives a homeomorphism of $\mathfrak{u}(V)$ with a neighbourhood of α in $U(V)$. It follows that $U(V)$ is a topological manifold.

Lens spaces

Now consider $C_n = \langle\omega\rangle < \mathbb{C}^\times$, where $\omega = e^{2\pi i/n}$.

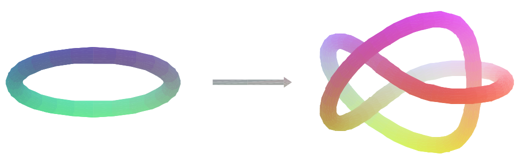
This acts by multiplication on $S(V)\simeq S(\mathbb{C}^d)\simeq S^{2d-1}$, so we can put $L = S(V)/C_n$.

Claim: L is a manifold of dimension $2d-1$.

To see this, let $\pi\colon S(V)\to S(V)/C_n$ be the projection map, and note that $\pi^{-1}\pi(U) = \bigcup_{k=0}^{d-1}\omega^k U$; this implies that π is an open map.
Next put $\epsilon = |\omega-1|/2$, and for $v\in S(V)$ put $N_\epsilon(v) = \{w\in S(V) \mid \|v-w\|<\epsilon\}$. One checks easily that $\|\omega^k u - u\|\geq 2\epsilon\|u\|$ and thus that $\pi\colon N_\epsilon(v)\to S(V)/C_n$ is injective.
It follows that $\pi\colon N_\epsilon(v)\to\pi N_\epsilon(v)$ is a homeomorphism and that the codomain is open in $S(V)$; this shows that $S(V)/C_n$ is a manifold.
We will see that $H^2(S(V)/C_n)\simeq\mathbb{Z}/n$. This is our first example where the cohomology is not a free abelian group.

Knot surgery

Let j be an embedding of the solid torus $S^1\times D^2$ in $\mathbb{R}^3\subset S^3$, whose image $K = j(S^1\times D^2)$ is a knot.



If we remove the interior of K we get a space X_0 with boundary $\partial(X_0) = S^1\times S^1$.

This is the same as the boundary of $X_1 = D^2\times S^1$.

We can thus glue X_0 and X_1 along their boundaries to get a new manifold called X. This is the most basic example of *surgery*: making new manifolds from old by cutting and gluing.

Cohomology of punctured euclidean space

- Consider a list $a_1,\dots,a_n$ of distinct points in $\mathbb{R}^d$ (with $d>1$) and put $M = \mathbb{R}^d\setminus\{a_1,\dots,a_n\}$.
- Define $f_i\colon M\to S^{d-1}$ by $f_i(x) = (x-a_i)/\|x-a_i\|$ and put $v_i = f_i^*(u_{d-1})\in H^{d-1}(M)$.
- As $u_{d-1}^2 = 0$ and f_i^* is a ring map we have $v_i^2 = 0$.
- Claim: we have $H^0(M) = \mathbb{Z}$ and $H^{d-1}(M) = \mathbb{Z}\{v_1,\dots,v_n\}$ and $H^k(M) = 0$ otherwise.
- For $n=0$ or $n=1$ we have seen this already.
- For $n>1$, put $A = \mathbb{R}^d\setminus\{a_1,\dots,a_{n-1}\}$ and $B = \mathbb{R}^d\setminus\{a_n\}$ so $M = A\cap B$ and $A\cup B = V$ (contractible).
- We have a Mayer-Vietoris sequence

$$0 = H^{d-1}(V)\to H^{d-1}(A)\oplus H^{d-1}(B)\to H^{d-1}(M)\xrightarrow{\delta} H^d(V) = 0,$$

  so $H^{d-1}(M)\simeq H^{d-1}(A)\oplus H^{d-1}(B)\simeq H^{d-1}(A)\oplus\mathbb{Z}.v_n$.
- A bit more work with the same Mayer-Vietoris sequence proves the full claim.
- In particular, $v_iv_j = 0$ for all i and j (because $H^{2d-2}(M) = 0$).

Cohomology of configuration space

- $F_n\mathbb{C} = \{z \in \mathbb{C}^n \mid z_p \neq z_q \text{ for all } p \neq q\}$.
- $f_{pq}\colon F_n\mathbb{C} \to S^1$ by $f_{pq}(z) = (z_q - z_p)/|z_q - z_p|$; $a_{pq} = f_{pq}^*(u_1) \in H^1(F_n\mathbb{C})$.
- Using $h(t,z) = e^{\pi i t} f_{pq}(z)$ we see that $f_{pq} \simeq f_{qp}$ and $a_{qp} = a_{pq}$.
- As f_{pq}^* is a ring map and $u_1^2 = 0$ we get $a_{pq}^2 = 0$.
- Define $g\colon F_3\mathbb{C} \to \mathbb{C} \times \mathbb{C}^\times \times (\mathbb{C} \setminus \{0,1\})$ by $g(z) = (z_0, z_1 - z_0, \frac{z_2 - z_0}{z_1 - z_0})$.
  This is a homeomorphism, with $g^{-1}(u,v,w) = (u, u+v, u+vw)$.
- Here $H^*(\mathbb{C}) = \mathbb{Z}$, $H^*(\mathbb{C}^\times) = \mathbb{Z}[u]/u^2$ and
  $H^*(\mathbb{C} \setminus \{0,1\}) = \mathbb{Z}[v_1, v_2]/(v_1^2, v_1 v_2, v_2^2)$.
- Thus, Künneth gives $H^*(F_3\mathbb{C}) = H^*(\mathbb{C}) \otimes H^*(\mathbb{C}^\times) \otimes H^*(\mathbb{C} \setminus \{0,1\}) =$
  $\mathbb{Z}[u, v_0, v_1]/(u^2, v_1^2, v_1 v_2, v_2^2) = \mathbb{Z}\{1, u, v_0, v_1, uv_0, uv_1\}$.
- One checks that $a_{01} = a_{10} = u$ and $a_{02} = a_{20} = u + v_0$ and
  $a_{12} = a_{21} = u + v_1$. It follows that
  $a_{01}a_{12} + a_{12}a_{20} + a_{20}a_{01} = u(u+v_1) + (u+v_1)(u+v_0) + (u+v_0)u =$
  $3u^2 + uv_1 + v_1 u + uv_0 + v_0 u = 0$.
- More generally, given distinct i, j, k we define $q\colon F_n\mathbb{C} \to F_3\mathbb{C}$ by
  $q(z) = (z_i, z_j, z_k)$, so $q^* a_{01} = a_{ij}$ and $q^* a_{12} = a_{jk}$ and $q^* a_{20} = a_{ki}$.
- By applying q^* to our relation in $H^*(F_3\mathbb{C})$ we get
  $a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij} = 0$ in $H^*(F_n\mathbb{C})$.
- Thus all the claimed relations are valid in $H^*(F_n\mathbb{C})$; we still need to check that there are no additional generators or relations.

Fibre bundle cohomology

- Consider a continuous map $p\colon E \to B$
  with fibres $F_b = p^{-1}\{b\}$ for $b \in B$ and inclusions $i_b\colon F_b \to E$.
- Suppose we have a basis $x_1, \dots, x_n$ for $H^*(B)$,
  and elements $y_1, \dots, y_m \in H^*(E)$ such that
  $i_b^*(y_1), \dots, i_b^*(y_m)$ is always a basis for $H^*(F_b)$.
- Expectation: $p^*(x_1)y_1, \dots, p^*(x_n)y_m$ should be a basis for $H^*(E)$.
- If $p = (B \times F \xrightarrow{\text{proj}} B)$ then this follows from the Künneth Theorem.
- More generally, it works for *fibre bundles*.
- Say $U \subseteq X$ is *even* if $(p^{-1}(U) \xrightarrow{p} U)$ is like $(U \times F \xrightarrow{\text{proj}} U)$.
- Say p is a *fibre bundle* if B can be covered by even open sets.
- Define $\phi_U\colon A(U)^* = \bigoplus_{i=1}^m H^{*-|y_i|}(U) \to B(U)^* = H^*(p^{-1}(U))$ by
  $\phi_U(a_1, \dots, a_m) = \sum_i p^*(a_i) y_i$.
- If U is even then ϕ_U is an isomorphism by Künneth.
- Claim: if U is even and ϕ_V is an isomorphism then so is $\phi_{U \cup V}$.
- If B is compact then $B = U_1 \cup \dots \cup U_p$ with U_i even and we conclude that ϕ_B is an isomorphism.
- This also works if B is not compact, by a limit argument.

The Five Lemma

Lemma
Suppose we have a commutative diagram with exact rows:

$$\begin{tikzcd}
A \arrow[r, "p"] \arrow[d, "\alpha", "\simeq"'] & B \arrow[r, "q"] \arrow[d, "\beta"', "\simeq"] & C \arrow[r, "r"] \arrow[d, "\gamma"] & D \arrow[r, "s"] \arrow[d, "\delta", "\simeq"'] & E \arrow[d, "\epsilon", "\simeq"'] \\
A' \arrow[r, "p'"'] & B' \arrow[r, "q'"'] & C' \arrow[r, "r'"'] & D' \arrow[r, "s'"'] & E'
\end{tikzcd}$$

If α, β, δ and ϵ are isomorphisms, then so is γ.

Proof Suppose that $c \in \ker(\gamma)$.
Then $\delta r(c) = r'\gamma(c) = 0$, but δ is iso, so $r(c) = 0$.
By exactness $c = q(b)$ for some $b \in B$.
Now $q'\beta(b) = \gamma q(b) = \gamma(c) = 0$.
By exactness $\beta(b) = p'(a')$ for some $a' \in A'$.
Put $a = \alpha^{-1}(a') \in A$, so $\beta p(a) = p'\alpha(a) = p'(a') = \beta(b)$
As β is iso, this gives $p(a) = b$.
We now have $c = q(b) = qp(a) = 0$, proving that γ is injective.
A similar type of argument proves surjectivity.

The induction step

$E \xrightarrow{p} B \qquad y_j \in H^*(E) \qquad A^k(U) = \bigoplus_j H^{k-|y_j|}(U) \xrightarrow{\phi_U} B^k(U) = H^*(p^{-1}(U))$
For each $b \in B$, the elements $i_b^*(y_j)$ give a basis of $H^*(F_b)$.

- For open sets $U, V \subseteq B$ we have Mayer-Vietoris sequences for (U, V) and for $(p^{-1}(U), p^{-1}(V))$ giving a diagram as follows:

$$\begin{tikzcd}
A^{k-1}U \times A^{k-1}V \arrow[r] \arrow[d, "\phi_U \times \phi_V"'] & A^{k-1}(U \cap V) \arrow[r] \arrow[d, "\phi_{U \cap V}"'] & A^k(U \cup V) \arrow[r] \arrow[d, "\phi_{U \cup V}"'] & A^k U \times A^k V \arrow[r] \arrow[d, "\phi_U \times \phi_V"] & A^k(U \cap V) \arrow[d, "\phi_{U \cap V}"] \\
B^{k-1}U \times B^{k-1}V \arrow[r] & B^{k-1}(U \cap V) \arrow[r] & B^k(U \cup V) \arrow[r] & B^k U \times B^k V \arrow[r] & B^k(U \cap V)
\end{tikzcd}$$

- If ϕ_U, ϕ_V and $\phi_{U \cap V}$ are isomorphisms, then so is $\phi_{U \cup V}$, by the Five Lemma.
- Suppose U is even and ϕ_V is iso. Then $U \cap V$ is also even so ϕ_U and $\phi_{U \cap V}$ are also iso, so $\phi_{U \cup V}$ is iso.
- Thus: if B can be covered by finitely many even open sets, then ϕ_B is iso.
- **Remark:** We have made the strong assumption that there are elements $y_j \in H^*(E)$ giving a basis for each $H^*(F_b)$. Without that assumption we need to use the Serre Spectral Sequence $H^i(B; H^j(F)) \Longrightarrow H^{i+j}(E)$ which is much more complicated.

Application to configuration space

- Define $p\colon F_{n+1}\mathbb{C} \to F_n\mathbb{C}$ by $p(z_0,\dots,z_n) = (z_0,\dots,z_{n-1})$.
- One can check that this is a fibre bundle.
- For $z = (z_0,\dots,z_{n-1}) \in F_n\mathbb{C}$ we have $p^{-1}\{z\} \simeq \mathbb{C}\setminus\{z_0,\dots,z_{n-1}\}$ so $H^*(p^{-1}\{z\}) = \mathbb{Z}\{1, v_0,\dots,v_{n-1}\} = \mathbb{Z}\{1, i^*(a_{0,n}),\dots,i^*(a_{n-1,n})\}$.
- Thus the fibre bundle theorem gives $H^i(F_{n+1}\mathbb{C}) = H^i(F_n\mathbb{C}) \oplus \bigoplus_{j=0}^{n-1} H^{i-1}(F_n\mathbb{C}).a_{jn}$
- From $F_3\mathbb{C} \simeq \mathbb{C}\times\mathbb{C}^\times \times (\mathbb{C}\setminus\{0,1\})$ we obtained $H^*(F_3\mathbb{C}) = \mathbb{Z}\{1, a_{01}, a_{02}, a_{12}, a_{01}a_{02}, a_{01}a_{12}\}$.
- It follows that the following set is a basis for $H^*(F_4\mathbb{C})$:

1	a_{01}	a_{02}	a_{12}	$a_{01}a_{02}$	$a_{01}a_{12}$
a_{03}	$a_{01}a_{03}$	$a_{02}a_{03}$	$a_{12}a_{03}$	$a_{01}a_{02}a_{03}$	$a_{01}a_{12}a_{03}$
a_{13}	$a_{01}a_{13}$	$a_{02}a_{13}$	$a_{12}a_{13}$	$a_{01}a_{02}a_{13}$	$a_{01}a_{12}a_{13}$
a_{23}	$a_{01}a_{23}$	$a_{02}a_{23}$	$a_{12}a_{23}$	$a_{01}a_{02}a_{23}$	$a_{01}a_{12}a_{23}$

- In particular, $H^*(F_4\mathbb{C})$ is generated as a ring by the elements a_{pq}.
- With a bit more pure algebra, we can also check that all relations follow from the relations $a_{pq} = a_{qp}$, $a_{pq}^2 = 0$ and $a_{pq}a_{qr} + a_{qr}a_{rp} + a_{rp}a_{pq} = 0$ mentioned previously.

Cohomology of complex projective space

- $\mathbb{C}P^n = \{[z] \mid z \in \mathbb{C}^{n+1}\setminus\{0\}\}$, where $[z] = [z']$ iff $z' \in \mathbb{C}^\times z$.
- $\mathbb{C}P^1 \simeq \mathbb{C}\cup\{\infty\} \simeq S^2$ by $[z_0 : z_1] \mapsto z_0/z_1$, so $H^*(\mathbb{C}P^1) = \mathbb{Z}\{1,x\} = \mathbb{Z}[x]/x^2$ with $x \in H^2(\mathbb{C}P^1)$.
- Claim: $H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/x^{n+1}$ with $x \in H^2(\mathbb{C}P^n)$.
- Or: $H^{2k}(\mathbb{C}P^n) = \mathbb{Z}.x^k$ for $0 \le k \le n$ but $H^j(\mathbb{C}P^n) = 0$ otherwise.
- Put $U = \{[z] \in \mathbb{C}P^n \mid z_n \neq 0\}$ and $V = \{[z] \in \mathbb{C}P^n \mid (z_0,\dots,z_{n-1}) \neq 0\}$.
- The map $[z] \mapsto (z_0,\dots,z_{n-1})/z_n$ gives $U \simeq \mathbb{C}^n$ and $U\cap V \simeq \mathbb{C}^n\setminus\{0\}$, so $H^*(U) = \mathbb{Z}$ and $H^*(U\cap V) = \mathbb{Z}\{1, u_{2n-1}\}$.
- We have $V \xrightarrow{r} \mathbb{C}P^{n-1} \xrightarrow{s} V$ by $r([z]) = [z_0,\dots,z_{n-1}]$ and $s([z_0,\dots,z_{n-1}]) = [z_0,\dots,z_{n-1},0]$. Clearly $rs = 1$, and using $h(t,[z]) = [z_0,\dots,z_{n-1},tz_n]$ we get $1 \simeq sr$. Thus $H^*(V) \simeq H^*(\mathbb{C}P^{n-1}) = \mathbb{Z}[x]/x^n$.
- For $p > 0$ we now have a Mayer-Vietoris sequence

$$H^{p-1}(\mathbb{C}P^{n-1}) \xrightarrow{k^*} H^{p-1}(S^{2n-1}) \xrightarrow{\delta} H^p(\mathbb{C}P^n) \xrightarrow{i^*} H^p(\mathbb{C}P^{n-1}) \xrightarrow{k^*} H^p(S^{2n-1})$$

  For most p the second and last terms are zero so $H^p(\mathbb{C}P^n) \simeq H^p(\mathbb{C}P^{n-1})$. In particular we have $x \in H^2(\mathbb{C}P^n)$ and $H^{2j}(\mathbb{C}P^n) = \mathbb{Z}.x^j$ for $0 \le j < n$.
- One exception is the case $p = 2n$ when we get $H^{2n}(\mathbb{C}P^n) = \mathbb{Z}.\delta(u_{2n-1})$. Different methods are needed to show that $\delta(u_{2n-1}) = \pm x^n$, completing the induction.

Cohomology of Milnor hypersurfaces

- Let $\mathbb{C}P^m \xleftarrow{p} \mathbb{C}P^m \times \mathbb{C}P^n \xrightarrow{q} \mathbb{C}P^n$ be the projection maps.
- We have seen that $H^*(\mathbb{C}P^m) = \mathbb{Z}[x]/x^{m+1}$ and $H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/x^{n+1}$.
- Put $y = p^*(x)$ and $z = q^*(x)$ so Künneth gives

$$H^*(\mathbb{C}P^m \times \mathbb{C}P^n) = \mathbb{Z}[y,z]/(y^{m+1}, z^{n+1}) = \mathbb{Z}\{y^iz^j \mid i \le m,\ j \le n\}.$$

- Now suppose that $m \le n$ and put $M = \text{Milnor hypersurface} = \{([z],[w]) \in \mathbb{C}P^m \times \mathbb{C}P^n \mid \sum_{i=0}^m z_iw_i = 0\}$. There are restricted projections $\mathbb{C}P^m \xleftarrow{p_1} M \xrightarrow{q_1} \mathbb{C}P^n$.
- $p_1^{-1}\{[z]\} = P(V_z)$, where $V_z = \{w \mid \sum_{i=0}^m z_iw_i = 0\}$, so $\{z^j \mid 0 \le j < n-1\}$ gives a basis for $H^*(p_1^{-1}\{[z]\})$.
- Fibre bundle theorem: $\{y^iz^j \mid i < m,\ j < n-1\}$ is a basis for $H^*(M)$.
- In particular z^{n-1} is expressible in terms of $1, z, \dots, z^{n-2}$.
- It turns out that

$$\begin{aligned} H^*(M) &= \mathbb{Z}[y,z]/(y^m, z^{n-1} - yz^{n-2} + \dots \pm y^{n-1}) \\ &= \mathbb{Z}\{y^iz^j \mid i \le m,\ j < n\} \end{aligned}$$

Cohomology of Fermat hypersurfaces

- Fix $d, n > 2$ and put $M = \text{Fermat hypersurface} = \{[z] \in \mathbb{C}P^{2n} \mid \sum_{k=0}^{2n} z_k^d = 0\}$.
- Claim: there are elements $x \in H^2(M)$ and $y \in H^{2n}(M)$ with $H^*(M) = \mathbb{Z}\{1, x, \dots, x^{n-1}, y, xy, \dots, x^{n-1}y\} = \mathbb{Z}[x,y]/(y^2, x^n - dy)$.
- Start of the proof: put $\omega = e^{i\pi/d}$ and define $j\colon \mathbb{C}P^{n-1} \xrightarrow{j} M \xrightarrow{r} \mathbb{C}P^{2n-1}$ by $j([z_0,\dots,z_{n-1}]) = [z_0,\dots,z_{n-1},\omega z_0,\dots,\omega z_{n-1},0]$.
- Also note that for $[z] \in M$ we have $(z_0,\dots,z_{2n-1}) \neq 0$ so can define $r\colon M \to \mathbb{C}P^{2n-1}$ by $r([z_0,\dots,z_{2n}]) = [z_0,\dots,z_{2n-1}]$.
- This gives $\mathbb{Z}[x]/x^{2n} \xrightarrow{r^*} H^*(M) \xrightarrow{j^*} \mathbb{Z}[x]/x^n$ with rj homotopic to the inclusion $\mathbb{C}P^{n-1} \to \mathbb{C}P^{2n-1}$ and so $j^*(r^*(x)) = x$.
- A typical point $[w] \in \mathbb{C}P^{2n-1}$ has preimage $r^{-1}\{[w]\} \subset M$ of size d (because a nonzero complex number has d different dth roots). From this we can deduce by degree theory that x^{2n-1} is divisible by d in $H^{4n-2}(M)$.
- Define $f\colon \mathbb{C}P^{2n} \to [0,1]$ by $f([z]) = |\sum_k z_k^d| / \sum_k |z_k^d|$, so $M = f^{-1}\{0\}$. We can try to deform $\mathbb{C}P^{2n}$ onto M by moving in the direction of steepest decrease of f. This fails because of stationary points, but the failure is controlled by Morse theory, which gives homological information.

Cohomology of flag spaces

- Recall that $\mathrm{Flag}(V)$ is the space of all lists $(W_0, \dots, W_d)$ where $0 = W_0 < W_1 < \dots < W_d = V$ with $\dim_{\mathbb{C}}(W_i) = i$.
- We can define $p_i \colon \mathrm{Flag}(V) \to PV$ by $p_i(W) = W_i \ominus W_{i-1}$ (the orthogonal complement of W_{i-1} in W_i). This gives $x_i = p_i^*(x) \in H^2(\mathrm{Flag}(V))$.
- Let s_k be the k'th elementary symmtric polynomial, i.e. the sum of all terms like $x_{i_1} \cdots x_{i_k}$ with $i_1 < \cdots < i_k$, or the coefficient of t^{d-k} in $\prod_i (t + x_i)$.
- We will show later that $H^*(\mathrm{Flag}(V)) = \mathbb{Z}[x_1, \dots, x_d]/(s_1, \dots, s_d)$.
- Let B be the set of monomials $x_1^{n_1} \dots x_d^{n_d}$ with $0 \leq n_i < i$ for all i; then B is a basis for $H^*(\mathrm{Flag}(V))$.
- To prove these statements, we will need to generalise them, to give statements that can be proved inductively using the fibre bundle theorem.

Relative and reduced cohomology

- For $Y \subseteq X$ put $C^*(X, Y) = \ker(i^* \colon C^*(X) \to C^*(Y))$ and $H^*(X, Y) = H^*(C^*(X, Y))$ (relative cohomology).
- This is a nonunital ring and a module over $H^*(X)$.
- The short exact sequence $C^*(X, Y) \to C^*(X) \to C^*(Y)$ gives a long exact sequence

$$H^{k-1}(Y) \xrightarrow{\delta} H^k(X, Y) \xrightarrow{\theta} H^k(X) \xrightarrow{i^*} H^k(Y) \xrightarrow{\delta} H^{k+1}(X, Y).$$

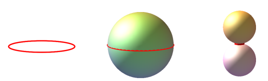
- $H^*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = H^*(B^n, S^{n-1}) = \mathbb{Z}.v_n$ where $v_n = \delta(u_{n-1}) \in H^n$.
- The maps δ and θ are $H^*(X)$-linear (with $\pm$-signs).
- If X has a specified basepoint $* \in X$ we put $\widetilde{C}^k(X) = C^k(X, \{*\})$ and

$$\widetilde{H}^k(X) = H^k(X, \{*\}) = \begin{cases} H^k(X) & \text{if } k > 0 \\ \{u \colon \pi_0(X) \to \mathbb{Z} \mid u(*) = 0\} & \text{if } k = 0 \end{cases}$$

- $\widetilde{H}^*(\mathbb{R}^n \setminus \{0\}) = \widetilde{H}^*(S^{n-1}) = \mathbb{Z}.u_{n-1}$.

Collapse and excision

- For closed $Y \subseteq X$ we let X/Y be the quotient space where Y is collapsed to a single point, taken as the basepoint.



S^1 S^2 S^2/S^1

- The collapse $p \colon X \to X/Y$ induces $p^* \colon \widetilde{H}^*(X/Y) \to H^*(X, Y)$, which is usually iso (when Y is closed).
- This works for submanifolds of manifolds, subcomplexes of simplicial complexes, subsets of $\mathbb{R}^n$ defined by polynomial inequalities.
- It can fail if X has an infinite amount of topological structure arbitrarily close to Y as with fractals.
- Keywords: *excision* and *neighbourhood deformation retract*.
- If $U \subseteq X$ is open we can often find $Y \subseteq U$ with Y closed in X such that $Y \to U$ is a homotopy equivalence;
  then $H^*(X, U) = H^*(X, Y)$, which is usually $\widetilde{H}^*(X/Y)$.
- Example: $H^{n-1}(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) = H^n(B^n, B^n \setminus \{0\}) = H^n(B^n, S^{n-1}) = \widetilde{H}^n(B^n/S^{n-1}) = \widetilde{H}^n(S^n) = \mathbb{Z}$.

Cohomology of the unitary group

- **Claim:** $H^*(U(n))$ is freely generated by elements $a_{2k-1} \in H^{2k-1}(U(n))$ for $1 \leq k \leq n$ with $a_i^2 = 0$.
- $H^*(U(3)) = E[a_1, a_3, a_5] = \mathbb{Z}\{1, a_1, a_3, a_5, a_1a_3, a_1a_5, a_3a_5, a_1a_3a_5\}$
- $U(1) = S^1$ and $U(2) = S^1 \times S^3$ by $(a, b, c) \mapsto \begin{bmatrix} ab & -\bar{c} \\ ac & \bar{b} \end{bmatrix}$.
  For $n > 2$ the spaces $U(n)$ and $P(n) = \prod_{k=1}^n S^{2k-1}$ have isomorphic cohomology rings but are not homotopy equivalent.
- Define $U(n) \xrightarrow{i} U(n+1) \xrightarrow{p} S^{2n+1}$ by

$$i(A) = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array}\right] \qquad p(B) = B.e_{n+1} = \text{ last column of } B.$$

- $p^{-1}\{e_{n+1}\} = i(U(n))$, and $p^{-1}\{u\} = B.i(U(n))$ for any B with $B.e_{n+1} = u$;
  so p is a fibre bundle projection.
- If we knew that $H^*(U(n)) = E[a_1, \dots, a_{2n-1}]$ and that there were elements $a_{2k-1} \in H^{2k-1}(U(n+1))$ for $k < n$ with $i^*(a_{2k-1}) = a_{2k-1}$ then we could put $a_{2n-1} = p^*(u_{2n-1})$ and the fibre bundle theorem would give $H^*(U(n+1)) = E[a_1, \dots, a_{2n+1}]$.

The complex reflection map

- For $z \in S^1$ and $L \in \mathbb{C}P^n$ we put $r(z, L) = z.1_L \oplus 1_{L^\perp}$ on $L \oplus L^\perp = \mathbb{C}^{n+1}$ or $r(z, [u]).v = v + (z - 1)\langle v, u\rangle u/\langle u, u\rangle \in v + L$.
- This gives a continuous map $r\colon S^1 \times \mathbb{C}P^n \to U(n+1)$.
- We also put $r(z, L, A) = r(z, L).A$ giving $r\colon S^1 \times \mathbb{C}P^n \times U(n) \to U(n+1)$.
- We will see that this is "almost a homeomorphism".
- Put $Y = (S^1 \times \mathbb{C}P^{n-1}) \cup (\{1\} \times \mathbb{C}P^n) \subset S^1 \times \mathbb{C}P^n$.
- For $z = 1$ we have $p(r(1, L)) = r(1, L).e_{n+1} = e_{n+1}$ always. For $z \neq 1$ we have $p(r(z, L)) = e_{n+1}$ iff $r(z, L).e_{n+1} = e_{n+1}$ iff $e_{n+1} \in L^\perp$ iff $L \in \mathbb{C}P^n$.
- Also, for $A \in U(n)$ we have $A.e_{n+1} = e_{n+1}$ so $p(r(z, L, A)) = p(r(z, L))$.
- Conclusion: $p(r(z, L, A)) = e_{n+1}$ iff $(z, L, A) \in Y \times U(n)$.
- Now consider $w \in S^{2n+1} \setminus \{e\}$ where $e = e_{n+1}$.
  Put $z = \langle w, w - e\rangle/\langle e, w - e\rangle$ and $L = \mathbb{C}.(w - e)$.
  Calculation gives $(z, L) \in (S^1 \times \mathbb{C}P^n) \setminus Y$ and $r^{-1}\{w\} = \{(z, L)\}$.
- Using this: r induces a homeomorphism
  $Q = (S^1 \times \mathbb{C}P^n \times U(n))/(Y \times U(n)) \to U(n+1)/U(n)$.
- Thus: a long exact sequence relates $H^*(U(n))$, $H^*(U(n+1))$ and $\widetilde{H}^*(Q)$.
- We will see that $\widetilde{H}^k(Q) = 0$ for $k < 2n+1$, so
  $H^k(U(n+1)) = H^k(U(n))$ for $k < 2n$.

Cohomology of the unitary group

- Recall $Y = (S^1 \times \mathbb{C}P^{n-1}) \cup (\{1\} \times \mathbb{C}P^n) \subset X = S^1 \times \mathbb{C}P^n$.
- Now $X \setminus Y = (S^1 \setminus \{1\}) \times (\mathbb{C}P^n \setminus \mathbb{C}P^{n-1})$
  and $S^1 \setminus \{1\} \simeq \mathbb{R}$ (stereographically)
  and $\mathbb{C}P^n \setminus \mathbb{C}P^{n-1} \simeq \mathbb{C}^n \simeq \mathbb{R}^{2n}$ (by $[z_0 : \ldots : z_n] \mapsto (z_0, \ldots, z_{n-1})/z_n$).
- Now $X \setminus Y \simeq \mathbb{R}^{2n+1}$ and $X/Y \simeq (X \setminus Y) \cup \{\infty\} \simeq \mathbb{R}^{2n+1} \cup \{\infty\} \simeq S^{2n+1}$.
- This gives $Q = (X \times U(n))/(Y \times U(n)) \simeq (S^{2n+1} \times U(n))/(\{*\} \times U(n))$
  so $\widetilde{H}^*(Q) = H^*(S^{2n+1} \times U(n), \{*\} \times U(n))$.
- Künneth gives $H^*(S^{2n+1} \times U(n)) = H^*(U(n)) \oplus H^*(U(n)).u_{2n+1}$.
- The LES for relative cohomology then gives
  $\widetilde{H}^*(Q) \simeq H^*(S^{2n+1} \times U(n), \{*\} \times U(n)) = H^*(U(n)).u_{2n+1}$.
- But also $Q \simeq U(n+1)/U(n)$ so $H^*(U(n+1), U(n)) \simeq H^*(U(n)).u_{2n+1}$.
- For $i < 2n$ we have $H^i(U(n+1), U(n)) = H^{i+1}(U(n+1), U(n)) = 0$ so
  $H^i(U(n+1)) \simeq H^i(U(n))$.
- Thus, for $k \leq n$ there is a unique $a_{2k-1} \in H^{2k-1}(U(n+1))$ that maps to $a_{2k-1} \in H^{2k-1}(U(n))$. We also put $a_{2n+1} = p^*(u_{2n+1}) \in H^{2n+1}(U(n+1))$.
- The restriction $i^*\colon H^*(U(n+1)) \to H^*(U(n))$ is a ring map that hits all the generators, so it is surjective. Thus $\delta = 0$ in the LES.
- We can now conclude that $H^*(U(n+1)) = E[a_1, \ldots, a_{2n+1}]$.

Hopf algebras

- Define $U(n)^2 \xrightarrow{\mu} U(n) \xleftarrow{\eta} 1$ by $\mu(A, B) = AB$ and $\eta(1) = I$.
  These make $U(n)$ a Lie group.
- Putting $A^* = H^*(U(n)) = E[a_1, a_3, \ldots, a_{2n-1}]$ we get ring maps
  $A^* \otimes A^* \xleftarrow{\psi = \mu^*} A^* \xrightarrow{\epsilon = \eta^*} \mathbb{Z}$.
- The associativity law says that $\mu(\mu \times 1) = \mu(1 \times \mu)\colon U(V)^3 \to U(V)$,
  and this implies that $(\psi \otimes 1)\psi = (1 \otimes \psi)\psi\colon A^* \to (A^*)^{\otimes 3}$.
  The unit laws imply that $(\epsilon \otimes 1)\psi = 1 = (1 \otimes \epsilon)\psi\colon A^* \to A^*$.

$$\begin{tikzcd} A^* & A^* \otimes A^* \arrow[l, "\epsilon\otimes 1"'] \arrow[r, "1\otimes\epsilon"] & A^* \\ & A^* \arrow[ul, "1"] \arrow[u, "\psi"] \arrow[ur, "1"'] & \end{tikzcd} \qquad \begin{tikzcd} A^* \arrow[r, "\psi"] \arrow[d, "\psi"'] & A^* \otimes A^* \arrow[d, "\psi\otimes 1"] \\ A^* \otimes A^* \arrow[r, "1\otimes\psi"'] & A^* \otimes A^* \otimes A^* \end{tikzcd}$$

- A structure like this is called a *Hopf algebra*.
- We say that $x \in A^n$ is *primitive* if $\epsilon(x) = 0$ and $\psi(x) = x \otimes 1 + 1 \otimes x$.
- For $A^* = H^*(U(n)) = E[a_1, \ldots, a_{2n-1}]$, the ring $A^* \otimes A^*$ is
  $E[b_1, \ldots, b_{2n-1}, c_1, \ldots, c_{2n-1}]$ where $b_{2i-1} = \pi_0^*(a_{2i-1})$, $c_{2i-1} = \pi_1^*(a_{2i-1})$.
- Claim: a_{2i-1} is primitive, i.e.
  $\mu^*(a_{2i-1}) = \pi_0^*(a_{2i-1}) + \pi_1^*(a_{2i-1}) = b_{2i-1} + c_{2i-1}$.
- Because μ^* is a ring map, this determines μ^* on all elements.

Proof of primitivity

- Claim: the map $\psi = \mu^*\colon E[a_1, \ldots, a_{2n-1}] \to E[b_1, \ldots, b_{2n-1}, c_1, \ldots, c_{2n-1}]$ sends a_{2i-1} to $b_{2i-1} + c_{2i-1}$.
- Put $u_{2i-1} = \psi(a_{2i-1}) - b_{2i-1} - c_{2i-1}$; we must show that $u_{2i-1} = 0$.
- From the counit laws $(\epsilon \otimes 1)\psi = (1 \otimes \epsilon)\psi = 1$ we see that
  $u_{2i-1} \in I^* \otimes I^*$ where $I^* = \ker(\epsilon) = \widetilde{H}^*(U(n))$.
- The inclusion $j\colon U(n-1) \to U(n)$ is a group homomorphism with
  $H^*(U(n-1)) = A^*/a_{2n-1}$; this gives a diagram

$$\begin{tikzcd} A^* \arrow[r, "\psi"] \arrow[d, "j^*"'] & A^* \otimes A^* \arrow[d, "(j\times j)^*"] \\ A^*/a_{2n-1} \arrow[r, "\psi"'] & (A^* \otimes A^*)/(b_{2n-1}, c_{2n-1}). \end{tikzcd}$$

- For $i < n$ we assume inductively that $j^*(a_{2i-1})$ is primitive; also
  $j^*(a_{2n-1}) = 0$ is primitive. So $u_{2i-1} \in J^* = (b_{2n-1}, c_{2n-1})$ for $i \leq n$.
- For $i < n$ we have $J^{2i-1} = 0$ so $u_{2i-1} = 0$.
- For $i = n$ we have $J^{2n-1} = \mathbb{Z}\{b_{2n-1}, c_{2n-1}\}$ but $I^* \otimes I^*$ is generated by all products $b_{2p-1}c_{2q-1}$ so $(I^* \otimes I^*) \cap J^*$ is zero in degree $2n - 1$.
  Thus u_{2n-1} is zero as well.

Vector bundles

- A *vector bundle* over a space X is a collection of finite-dimensional vector spaces V_x for each $x \in X$, "varying continuously".
- There must be a given topology on the total space $EV = \{(x,v) \mid x \in X,\ v \in V_x\}$ such that $p\colon (x,v) \mapsto x$ is continuous.
- Say $U \subseteq X$ is *even* if there is a homeomorphism $p^{-1}(U) \simeq \mathbb{R}^d \times U$ compatible with projection and vector space structure.
- We require that X can be covered by even open sets.
- We usually assume that X is compact.
- It is harmless to assume that there are continuously varying inner products.
- Example: for $z \in S^1$ put $V_z = \{w \in \mathbb{C} \mid w^2 \in \mathbb{R}_+ z\}$ so $V_{\exp(i\theta)} = \mathbb{R}.\exp(i\theta/2)$. This is a vector bundle, and EV is a Möbius strip.
- The *tangent bundle* of S^n is $T_x S^n = \{v \in \mathbb{R}^{n+1} \mid \langle x, v\rangle = 0\}$.
- The *tautological bundle* over $\mathbb{C}P^n$ is $T_L = L$, so $ET = \{(v,L) \mid v \in L,\ L \leq \mathbb{C}^{n+1},\ \dim(L) = 1\}$.
- The *image bundle* over $P = \{A \in M_n(\mathbb{C}) \mid A^2 = A\}$ is $W_A = \operatorname{img}(A) = \ker(I - A)$.
- Many interesting spaces can be described in terms of vector bundles.

Thom spaces

- If V is a d-dimensional vector bundle over a compact space X we define the Thom space X^V as $EV \cup \{\infty\}$.
- We will prove the Thom Isomorphism Theorem: if V is oriented then $\widetilde{H}^k(X^V) \simeq H^{k-d}(X)$.
- Many calculations can be deduced from this.
- Recall the Möbius bundle $V_{\exp(i\theta)} = \mathbb{R}.\exp(i\theta/2)$ over S^1. Define $f\colon EV \to \mathbb{R}P^2 = (\mathbb{R}^3 \setminus \{0\})/\mathbb{R}^\times$ by $f(e^{i\theta}, te^{i\theta/2}) = [\cos(\theta/2), \sin(\theta/2), t]$. With $f(\infty) = [0,0,1]$ this gives $(S^1)^V \simeq \mathbb{R}P^2$.
- Recall the tautological bundle T over $\mathbb{C}P^n$ with $T_L = L$. One can check that there is a well-defined $f\colon ET \to \mathbb{C}P^{n+1}$ given by $f(v, \mathbb{C}u) = \mathbb{C}.(u, \langle u, v\rangle)$. With $f(\infty) = \mathbb{C}.e_{n+1}$ this gives $(\mathbb{C}P^n)^T \simeq \mathbb{C}P^{n+1}$.
- After choosing inner products we can put $B(V) = \{(x,v) \in EV \mid \|x\| \leq 1\}$ and $S(V) = \{(x,v) \in EV \mid \|x\| = 1\}$ and $EV^\times = \{(x,v) \in EV \mid v \neq 0\}$
- Recall that $\mathbb{R}^d \cup \{\infty\} \simeq S^d \simeq B^d/S^{d-1}$. By doing this in each fibre we get $X^V \simeq B(V)/S(V)$.
- This gives $\widetilde{H}^*(X^V) = H^*(B(V), S(V)) = H^*(EV, EV^\times)$.

Orientations and Thom classes

- For a vector space $V \simeq \mathbb{R}^d$, let $\operatorname{Or}(V)$ be the set of generators of the group $H^d(V, V^\times) \simeq \mathbb{Z}$ (so $|\operatorname{Or}(V)| = 2$).
- If V is a complex vector space, there is a canonical orientation (because $GL_n(\mathbb{C})$ is connected).
- If V is a d-dimensional vector bundle over X, the set $\operatorname{Or}(X) = \{(x,u) \mid x \in X,\ u \in \operatorname{Or}(V_x)\}$ has a natural topology as a double cover of X.
- An *orientation* of V is a continuous choice of $u_x \in \operatorname{Or}(V_x)$ for each $x \in X$.
- The Möbius bundle has no orientation; but any complex bundle has a canonical orientation.
- A *Thom class* for V is an element $u \in H^d(EV, EV^\times)$ such that $i_x^*(u) \in H^d(V_x, V_x^\times)$ is a generator for all $x \in X$.
- **Theorem (Thom):** there is a natural bijection from Thom classes to orientations. Moreover, if u is a Thom class then multiplication by u gives an isomorphism $H^k(X) \to H^{k+d}(EV, EV^\times) \simeq \widetilde{H}^{k+d}(X^V)$.
- The proof is like the fibre bundle theorem. If U_0, U_1 are open in X, and the claim holds for U_0, and U_1 is even, then the claim holds for $U_0 \cup U_1$ by a Mayer-Vietoris sequence. The claim therefore holds for finite unions of even sets, and thus for compact subsets of X.

The Euler class

- Let V be an oriented n-dimensional vector bundle over X, with Thom class $u(V) \in \widetilde{H}^n(X^V)$.
- Define $i\colon X \to X^V$ by $i(x) = 0 \in V_x \subset X^V$.
- Put $e(V) = i^*(u(V)) \in H^n(X)$. This is called the *Euler class* of V.
- If $EV = \mathbb{R} \times X$ then i is homotopic to the constant map at ∞ and so $e(V) = 0$.
- In general one can show that $e(U \oplus W) = e(U)e(W)$.
- So if $V \simeq \mathbb{R} \oplus W$ then $e(V) = 0$.
- A *section* of V is a continuous map $s\colon X \to EV$ with $s(x) \in V_x$ for all x.
- If s is a section with $s(x) \neq 0$ for all x then we can put $U_x = \mathbb{R}.s(x)$ and $W_x = U_x^\perp$ to get $e(V) = 0$.
- By contrapositive: if $e(V) \neq 0$ then every section of V must vanish somewhere.
- Later we will see other *characteristic classes* giving invariants in $H^*(X)$ of vector bundles over X; these help to classify vector bundles up to isomorphism.

The Gysin sequence

- Let V be an oriented n-dimensional vector bundle over X, with Euler class $e(V) \in H^n(X)$.
- The pair (BV, SV) has a long exact sequence:

$$\dots \to H^k(BV, SV) \xrightarrow{\alpha} H^k(BV) \xrightarrow{\beta} H^k(SV) \xrightarrow{\delta} H^{k+1}(BV, SV) \to \dots.$$

- Here $H^k(BV, SV) = \widetilde{H}^k(BV/SV) = \widetilde{H}^k(X^V) = H^{k-n}(X).u(V)$.
- The projection $p\colon BV \to X$ is a homotopy equivalence, with inverse given by the zero section $X \to BV$; so $H^k(BV) = H^k(X)$.
- This identifies α with the map i^* so $\alpha(a) = a.e(V)$.
- We now have an exact sequence as follows, called the *Gysin sequence*.

$$\to H^{k-1}(SV) \to H^{k-n}(X) \xrightarrow{\times e(V)} H^k(X) \xrightarrow{\beta} H^k(SV) \xrightarrow{\delta} H^{k+1-n}(X) \to \dots.$$

- **Example:** for the tautological bundle T over $\mathbb{C}P^n$ we have $ST = \{(v, L) | v \in L \leq \mathbb{C}^{n+1}, \|v\| = 1\} = \{(v, \mathbb{C}v) | v \in \mathbb{C}^{n+1}, \|v\| = 1\} \simeq S^{2n+1}$. Also $e(T) = x$ and H^*ST is mostly zero so $\times x\colon H^{k-2}\mathbb{C}P^n \to H^k\mathbb{C}P^n$ is usually iso. Now we can complete the proof that $H^*(\mathbb{C}P^n) \simeq \mathbb{Z}[x]/x^{n+1}$.

Classifying vector bundles

- $\mathrm{Vect}_k(X) = \{$iso classes of k-dimensional vector bundles over $X\}$
- $\mathrm{Vect}(X)$ is a semiring with $[U] + [V] = [U \oplus V]$, $[U][V] = [U \otimes V]$. This is commutative but there are no additive inverses.
- **Theorem:** there are spaces G_k with $\mathrm{Vect}_k(X) \simeq [X, G_k]$ for all compact X.
- Put $P = \mathbb{R}[t]$ and $P_m = \{f \in P \mid \deg(f) < m\}$. Put $G_{km} = \{V \leq P_m \mid \dim(V) = k\}$ and $G_k = \{V \leq P \mid \dim(V) = k\} = \bigcup_m G_{km}$.
- Define $\theta_{km} \colon \widetilde{G}_{km} = \{$injective linear $\alpha\colon \mathbb{R}^k \to P_m\} \to G_{km}$ by $\theta(\alpha) = \alpha(\mathbb{R}^k)$. Declare that $U \subseteq G_k$ is open iff $\theta_{km}^{-1}(U)$ is open for all k and m.
- Define a tautological bundle T over G_k by $ET = \{(v, V) \mid v \in V \in G_k\}$.
- For any $f\colon X \to Y$ and any W over Y, define $f^*(W)_x = W_{f(x)}$ so $E(f^*W) = \{(x, w) \in X \times EW \mid f(x) = \pi(w)\}$.
- We now have $\phi_0\colon \mathrm{Map}(X, G_k) \to \mathrm{Vect}_k(X)$ by $\phi_0(f) = [f^*(T)]$.
- **Claim:** every V over X is isomorphic to $f^*(T)$ for some f, and $f_0^*(T) \simeq f_1^*(T)$ iff f_0 and f_1 are homotopic.

Partitions of unity

- Let X be a compact Hausdorff space with an open cover $\mathcal{U} = (U_i)_{i \in I}$.
- For $\phi\colon X \to [0, \infty)$ we put $\mathrm{supp}(\phi) = \overline{\phi^{-1}((0, \infty))}$.
- A *partition of unity* subordinate to $\mathcal{U}$ is a list $\phi_1, \dots, \phi_n\colon X \to [0, 1]$ with $\sum_j \phi_j = 1$ such that for each j there exists i with $\mathrm{supp}(\phi_j) \subseteq U_i$.
- **Lemma:** there always exists a partition of unity.
- **Proof:** For each x choose i with $x \in U_i$.
- By standard general topology and Urysohn's Lemma: we can choose $\psi_x\colon X \to [0, 1]$ with $\psi_x(x) = 1$ and $\mathrm{supp}(\psi_x) \subseteq U_i$.
- The open sets $V_x = \psi_x^{-1}((0, \infty))$ cover the compact space X, so we can choose $x_1, \dots, x_n$ with $\bigcup_{j=1}^n V_{x_j} = X$.
- Now put $\psi = \sum_{j=1}^n \psi_{x_j}$ so $\psi > 0$ everywhere. Put $\phi_j = \psi_{x_j}/\psi$. $\square$
- **Example:** let V be a vector bundle over X, and let $\mathcal{U}$ be the family of even open sets, i.e. those over which EV looks like $\mathbb{R}^d \times U$. Then there are maps $\phi_1, \dots, \phi_n\colon X \to [0, 1]$ and even open sets $U_1, \dots, U_n$ with $\sum_j \phi_j = 1$ and $\mathrm{supp}(\phi_j) \subseteq U_j$.
- By adjusting the argument slightly we can assume that there are even open sets U_j' with $\overline{U_j} \subseteq U_j'$.

Extension of sections

- Let Y be a closed subset of a compact Hausdorff space X.
- **Tietze's Theorem:** any continuous map $Y \to \mathbb{R}$ can be extended to a continuous map $X \to \mathbb{R}$.
- Let V be a vector bundle over X. A section of V over Y is a continuous map $s\colon Y \to EV$ with $\pi(s(y)) = y$ (i.e. $s(y) \in V_y$) for all y.
- **Theorem:** any section s over Y can be extended over X.
- **Proof:** first suppose that V is constant, so $EV = \mathbb{R}^d \times X$ and sections over Y are just maps $Y \to \mathbb{R}^d$. This case is immediate from Tietze's theorem.
- More generally, choose ϕ_j, U_j, U_j' where $\mathrm{supp}(\phi_j) \subseteq U_j \subseteq \overline{U_j} \subseteq U_j'$ and V is constant over U_j'. By the previous case we can choose s_j over $\overline{U_j}$ extending $s|_{Y \cap \overline{U_j}}$.
- Define t_j to be $\phi_j s_j$ on $\overline{U_j}$, and 0 outside U_j. As $\mathrm{supp}(\phi_j) \subseteq U_j$ this definition is consistent and gives a continuous section.
- Define a section $t = \sum_j t_j$ over X; as $\sum_j \phi_j = 1$ this extends s. $\square$
- **Application:** A morphism $\alpha\colon V \to W$ is the same as a section of the bundle $\mathrm{Hom}(V, W)_x = \mathrm{Hom}(V_x, W_x)$. Thus, if we have a morphism defined only over Y, we can extend it to get a morphism defined over X.

The isomorphism locus is open, and homotopy invariance

- Let $\alpha\colon V \to W$ be a morphism of d-dimensional vector bundles over X, and put $A = \{x \mid \alpha_x\colon V_x \to W_x \text{ is iso }\}$.
- **Claim:** A is open.
- First suppose that V and W are constant, so $EV = EW = \mathbb{R}^d \times X$. Then α is essentially a continuous map $X \to \mathrm{Hom}(\mathbb{R}^d, \mathbb{R}^d) = M_d(\mathbb{R})$, and $A = \{x \mid \det(\alpha_x) \neq 0\}$, which is open.
- In general, for any $x \in X$ we can choose an open neighbourhood U on which V and W are constant. The previous case then shows that $A \cap U$ is open. As this works for all x we see that A is open.
- **Corollary:** if $f_0 \simeq f_1$ via $h\colon [0,1] \times X \to Y$ then $f_0^*(W) \simeq f_1^*(W)$.
- For $a \in [0,1]$ define $(V_a)_x = W_{h(a,x)}$; we must show that $V_0 \simeq V_1$. Write $a \sim b$ if $V_a \simeq V_b$. If the equivalence classes are open, connectedness of $[0,1]$ implies that $0 \sim 1$.
- Define U, U' over $[0,1] \times X$ by $U_{(t,x)} = W_{h(t,x)}$ and $U'_{(t,x)} = W_{h(a,x)}$.
- The identity gives an isomorphism $\alpha\colon U \to U'$ over $\{a\} \times X$. By the section extension lemma, this can be extended to a homomorphism $\alpha\colon U \to U'$ over all of $[0,1] \times X$.
- The invertibility locus of α is open and contains $\{a\} \times X$. As X is compact it contains some $(a-\epsilon, a+\epsilon) \times X$, so the equivalence class of a contains $(a-\epsilon, a+\epsilon)$. $\square$

Global generation

- Let V be a d-dimensional vector bundle over X. **Claim:** for some N there is a map $\alpha\colon \mathbb{R}^N \times X \to EV$ that is a linear surjection on each fibre.
- **Proof:** as before we can find open sets $U_1, \dots, U_n, U'_1, \dots, U'_n$ with $X = U_1 \cup \dots \cup U_n$ and $\overline{U_i} \subseteq U'_i$ and U'_i is even.
- As U'_i is even and contains $\overline{U_i}$, we can choose an isomorphism $\alpha_i\colon \mathbb{R}^d \to V$ over $\overline{U_i}$, and then extend it to get a homomorphism $\alpha_i\colon \mathbb{R}^d \to V$ over all of X.
- Now define $\alpha\colon \mathbb{R}^{dn} \to V$ by $\alpha(u_1, \dots, u_n) = \sum_i \alpha_i(u_i)$. Over $\overline{U_i}$ we know that α_i is iso so α is surjective. As $X = \bigcup_i U_i$ it follows that α is surjective everywhere. $\square$
- **Corollary:** there is a map $f\colon X \to G_d$ with $V \simeq f^*(T)$.
- **Proof:** Choose α as before, so $\alpha_x\colon P_N = \mathbb{R}^N \to V_x$ is a linear surjection. It follows that $\dim(\ker(\alpha_x)) = N - d$ and $\dim(\ker(\alpha_x)^\perp) = d$, so we can define $f\colon X \to G_{dN} \subset G_d$ by $f(x) = \ker(\alpha_x)^\perp$.
- It is easy to see that α_x restricts to give an isomorphism $(f^*(T))_x = \ker(\alpha_x)^\perp \to V_x$, so $f^*(T) \simeq V$. $\square$

Isomorphism implies homotopy

- Suppose $f_0, f_1\colon X \to G_d$ with $f_0^*(T) \simeq f_1^*(T)$. **Claim:** $f_0 \simeq f_1$.
- **Proof:** As X is compact, so is $f_0(X) \cup f_1(X)$. It follows (by a lemma) that $f_0(X) \cup f_1(X) \subseteq G_{dN}$ for some N, i.e. $f_i(x) \leq P_N \subset P = \mathbb{R}[t]$.
- By assumption we have isomorphisms $\alpha(x)\colon (f_0^* T)_x = f_0(x) \to (f_1^* T)_x = f_1(x)$ for all x.
- For $s \in [0,1]$ and $x \in X$ define $\beta(s,x)\colon f_0(x) \to P_{2N} = P_N \oplus t^N P_N$ by $\beta(s,x)(v) = (1-s)v + st^N \alpha(x)(v)$. This is injective so we can define $h(s,x) = \beta(s,x)(f_0(x)) \in G_k$.
- This gives a homotopy from f_0 to $t^N f_1$.
- We can also define $\gamma(s,x)\colon t^N f_1(x) \to P_{2N}$ by $\gamma(s,x)(t^N v) = sv + (1-s)t^N v$, and then $k(s,x) = \gamma(s,x)(t^N f_1(x)) \in G_k$. This gives a homotopy from $t^N f_1$ to f_1. $\square$
- Homotopy invariance means we can define $\phi\colon [X, G_k] \to \mathrm{Vect}_k(X)$ by $\phi([f]) = [f^*(T)]$. Global generation allowed us to prove that ϕ is surjective. This slide shows that ϕ is injective. We therefore have a natural bijection $\phi\colon [X, G_k] \to \mathrm{Vect}_k(X)$.
- There is a similar result for complex vector bundles and the space of k-dimensional complex subspaces of $\mathbb{C}[t]$.

Classifying line bundles

- From now on everything is over $\mathbb{C}$ by default.
- $\mathrm{Pic}(X) = \mathrm{Vect}_1(X) = [X, G_1] = [X, \mathbb{C}P^\infty]$
- This is a group with $[L][M] = [L \otimes M]$ and $1 = [\mathbb{C}]$ and $[L]^{-1} = [L^*] = [\mathrm{Hom}(L, \mathbb{C})]$ (because $L \otimes L^* \simeq \mathbb{C}$).
- Recall $G_1 = \mathbb{C}P^\infty = \{L < \mathbb{C}[t] \mid \dim(L) = 1\}$.
- Multiplication $\mu\colon \mathbb{C}[t] \times \mathbb{C}[t] \to \mathbb{C}[t]$ induces $\mu\colon \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ and then $\mu\colon [X, \mathbb{C}P^\infty] \times [X, \mathbb{C}P^\infty] \to [X, \mathbb{C}P^\infty]$. This is the same product operation as before.
- We have seen that $H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/x^{n+1}$ with $x = e(T)$. It is also true that $H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x]$ with $x = e(T)$.
- For a line bundle L over X we have $e(L) \in H^2(X)$. If $L \simeq f^*(T)$ for some $f\colon X \to \mathbb{C}P^\infty$ then $e(L) = e(f^*(T)) = f^*(e(T)) = f^*(x)$.
- Note that $\mu^*(x) \in H^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \mathbb{Z}\{x \otimes 1, 1 \otimes x\}$ and $\mu^*(x)$ restricts to x on $\mathbb{C}P^\infty \times \{1\}$ or $\{1\} \times \mathbb{C}P^\infty$.
- From this: $\mu^*(x) = x \otimes 1 + 1 \otimes x$, and then $e(L \otimes M) = e(L) + e(M)$.
- We now have a group homomorphism $e\colon \mathrm{Pic}(X) \to H^2(X)$. It can be shown that this is an isomorphism.

Cohomology of projective bundles

- Let V be a complex vector bundle of dimension d over X.
- Define $PV = \{(a, L) \mid a \in X,\ L \leq V_a,\ \dim(L) = 1\} = \coprod_a P(V_a)$.
- This has a natural topology making it a fibre bundle over X, with fibres $P(V_a)$ homeomorphic to $\mathbb{C}P^{d-1}$.
- Define a tautological bundle T over PV by $T_{(a,L)} = L$ or $ET = \{(a, L, v) \mid a \in X,\ L \leq V_a,\ \dim(L) = 1,\ v \in V_a\}$.
- This gives an element $e(T) \in H^2(PV)$ which we also call x. Put $B = (1, x, \dots, x^{d-1})$.
- For $a \in X$ we have $i_a \colon P(V_a) \to PV$ with $i_a^*(T) = T$ and $i_a^*(x) = x \in H^2(P(V_a))$ so $i_a^*(B)$ is a basis for $H^*(P(V_a))$.
- By the fibre bundle theorem: B is a basis for $H^*(PV)$ over $H^*(X)$.
- Although $-x^d$ maps to zero on each fibre, it does not follow that $-x^d = 0$.
- Instead: we can express $-x^d$ in terms of B, so there are unique elements $c_i(V) \in H^{2i}(X)$ with $x^d + c_1(V)x^{d-1} + \cdots + c_{d-1}(V)x + c_d(V) = 0$.
- We put $c_0(V) = 1$ and $f_V(t) = \sum_{i=0}^d c_i(V)t^{d-i}$ so $f_V(x) = 0$ and $H^*(PV) \simeq H^*(X)[t]/f_V(t)$.
- The $c_i(V)$ are *Chern classes* and $f_V(t)$ is the *Chern polynomial*.

Chern classes of sums

- Consider complex vector spaces V, W.
  Suppose that $L \leq V \oplus W$ is one-dimensional and $L \not\leq W$.
  The projection $\pi \colon V \oplus W \to V$ gives an isomorphism $L \to \pi(L)$.
  Composing the inverse with $\pi' \colon V \oplus W \to W$ gives $\alpha \colon \pi(L) \to W$.
- From this we get $P(V \oplus W) \setminus P(W) \simeq E(\mathrm{Hom}(T, W))$ and $P(V \oplus W)/P(W) \simeq P(V)^{\mathrm{Hom}(T,W)}$.
- This also works for vector bundles and projective bundles.
- This gives an LES relating $H^*(P(V \oplus W))$ and $H^*(P(W))$ and $\widetilde{H}^*(P(V)^{\mathrm{Hom}(T,W)}) \simeq H^{*-2\dim(W)}(P(V))$; similarly with V, W exchanged.
- Here $H^*(P(V \oplus W)) = H^*(X)[t]/f_{V \oplus W}(t)$; similarly for $P(V)$ and $P(W)$.
- From this we can prove $f_{V \oplus W}(t) = f_V(t)f_W(t)$.
- Equivalently $c_k(V \oplus W) = \sum_{k=i+j} c_i(V)c_j(W)$.
- If $\dim(V) = d$ then $c_d(V) = (-1)^d e(V)$; so for line bundles $c_1(L) = -e(L) = e(L^*)$ and $f_L(t) = t - e(L)$.
- So if $V \simeq L_1 \oplus \cdots \oplus L_d$ then $f_V(t) = \prod_i (t - e(L_i))$.
- So if V is the constant bundle $\mathbb{C}^d$ then $f_V(t) = t^d$, $c_k(V) = 0$ for $k > 0$.

Relations for flag manifolds

- Recall that $\mathrm{Flag}(\mathbb{C}^n)$ is the space of flags $W = (W_0 < W_1 < \cdots < W_n = \mathbb{C}^n)$ with $\dim(W_i) = i$.
- We have a line bundle L_i over $\mathrm{Flag}(\mathbb{C}^n)$ with $(L_i)_W = W_i/W_{i-1}$. This gives elements $x_i = e(L_i) \in H^2(\mathrm{Flag}(\mathbb{C}^n))$ for $i = 1, \dots, n$, with $f_{L_i}(t) = t - x_i$.
- If we put $V = \bigoplus_i L_i$ we get $f_V(t) = \prod_i (t - x_i)$, so $c_k(V) = \pm\sigma_k$, where σ_k is the k'th elementary symmetric function of the variables x_i.
- The inner product gives a splitting $\mathbb{C}^n = W_n \simeq \bigoplus_{i=1}^n (W_i/W_{i-1})$ so $V = \bigoplus_i L_i \simeq \mathbb{C}^n$ as bundles so $f_V(t) = t^n$.
- It follows that $\sigma_k = 0$ for $1 \leq k \leq n$.
- In fact $H^*(\mathrm{Flag}(\mathbb{C}^n)) = \mathbb{Z}[x_1, \dots, x_n]/(\sigma_1, \dots, \sigma_n)$; to be proved later.
- **Example:** $H^* \mathrm{Flag}(\mathbb{C}^3) = \mathbb{Z}[x, y, z]/(x + y + z,\ xy + xz + yz,\ xyz)$.
- First relation gives $z = -x - y$; substitute in other relations to get $x^2 + xy + y^2 = 0$, $x^2y + xy^2 = 0$.
- Second relation now gives $y^2 = -x^2 - xy$; substitute in third to get $x^3 = 0$.
- Now $H^* = \mathbb{Z}[x, y, z]/(x^3 = 0, y^2 = -x^2 - xz, z = -x - y0) = \mathbb{Z}\{1, x, x^3, y, xy, x^2y\}$.

Milnor hypersurfaces revisited

- Recall that for $m \leq n$ we defined $H_{m,n} = \{([z], [w]) \in \mathbb{C}P^m \times \mathbb{C}P^n \mid \sum_{i=0}^m z_i w_i = 0\}$.
- This has projections $\mathbb{C}P^m \xleftarrow{p} H_{m,n} \xrightarrow{q} \mathbb{C}P^n$ and we put $y = p^*(x)$, $z = q^*(x) \in H^2(H_{m,n})$.
- Define a bundle V over $\mathbb{C}P^m$ by $W_{[z]} = \{w \in \mathbb{C}^{n+1} \mid \sum_{i=0}^m z_i w_i = 0\}$.
  Then $H_{m,n} = PV$ and so
  $H^*(H_{m,n}) = H^*(\mathbb{C}P^n)\{z^j \mid 0 \leq j < n\} = \mathbb{Z}[y, z]/(y^{m+1}, f_V(z))$.
- For $L \in \mathbb{C}P^m$ we define $\alpha_L \colon \mathbb{C}^{n+1} \to L^*$ by $\alpha(w)(v) = \sum_{i=0}^m w_i v_i$. This gives a surjective map $\alpha \colon \mathbb{C}^n \to T^*$ of vector bundles with $\ker(\alpha) = V$.
- Using the inner product we get $T^* \oplus V \simeq \mathbb{C}^{n+1}$ so $(t + y)f_V(t) = f_{T^*}(t)f_V(t) = f_{\mathbb{C}^{n+1}}(t) = t^{n+1}$ in $H^*(\mathbb{C}P^m)[t] = \mathbb{Z}[y, t]/y^{m+1}$.
- By long division we get $f_V(t) = t^n - yt^{n-1} + y^2t^{n-2} - \cdots \pm y^m t^{n-m}$. Thus in $H^*(H_{m,n})$ we have $\sum_{i=0}^m (-1)^i y^i z^{n-i} = f_V(z) = 0$.

Cohomology of Lens spaces

- Put $\omega = e^{2\pi i/n}$ and $C_n = \langle\omega\rangle < \mathbb{C}^\times$ so C acts by multiplication on $S(\mathbb{C}^{d+1}) = S^{2d+1}$. Put $M = S^{2d+1}/C_n$ (a Lens space).
- Let T = tautological bundle over $\mathbb{C}P^d$, so $e(T) = x$ and $e(T^{\otimes n}) = nx$.
- Define $\phi\colon S^{2d+1} \to S(T^{\otimes n}) = \{(L, v) \mid L \in \mathbb{C}P^d,\ v \in L^{\otimes n},\ \|v\| = 1\}$ by $\phi(u) = (\mathbb{C}u, u^{\otimes n})$.
- Then ϕ is surjective and $\phi(u) = \phi(u')$ iff $u' = \omega^k u$ for some k.
- Thus ϕ induces a homeomorphism $M = S^{2d+1}/C_n \to S(T^n)$.
- This gives a Gysin sequence $H^{k-2}(\mathbb{C}P^d) \xrightarrow{\times nx} H^k(\mathbb{C}P^d) \to H^k(M) \xrightarrow{\delta} H^{k-1}(\mathbb{C}P^d) \xrightarrow{\times nx} H^{k+1}(\mathbb{C}P^d)$.
- This gives a short exact sequence $\mathbb{Z}[x]/(x^{d+1}, nx) = H^*(\mathbb{C}P^d)/nx \to H^*(M) \to \mathrm{ann}(nx, H^*(\mathbb{C}P^d)) = \mathbb{Z}.x^d$
- This gives $H^*(M) = \mathbb{Z}[x]/(x^{d+1}, nx) \oplus \mathbb{Z}v$ with $|v| = 2d + 1 = \dim(M)$.
- **Example:** For $d = 4$ we have $H^*(M) = (\mathbb{Z}, 0, (\mathbb{Z}/n)x, 0, (\mathbb{Z}/n)x^2, 0, (\mathbb{Z}/n)x^3, 0, (\mathbb{Z}/n)x^4, \mathbb{Z}v, 0, 0, \dots)$.

Ring structure of cohomology of flag bundles

- $\mathrm{Flag}_1(V) = PV$, so $H^*(\mathrm{Flag}_1(V)) = H^*(X)[x_1]/f_V(x_1)$.
- As $f_V(x_1) = 0$, we have $f_V(t) = (t - x_1)g_1(t)$ for some monic $g_1(t) \in H^*(\mathrm{Flag}_1(V))[t]$ of degree $d - 1$.
- As $U_0 = V = L_1 \oplus U_1$ we have $f_V(t) = (t - x_1)f_{U_1}(t)$ so $g_1(t) = f_{U_1}(t)$.
- As $\mathrm{Flag}_2(V) = P(U_1)$ we have $H^*(\mathrm{Flag}_2(V)) = H^*(\mathrm{Flag}_1(V))[x_2]/f_{U_1}(x_2)$.
- This is also $H^*(X)[x_1, x_2]/(g_0(x_1), g_1(x_2))$, where $g_0(t) = f_V(t)$ and $g_1(t) = f_V(t)/(t - x_1)$.
- In general $H^*(\mathrm{Flag}_k(V)) = H^*(X)[x_1, \dots, x_k]/(g_{i-1}(x_i) \mid 1 \le i \le k)$ where $g_0(t) = f_V(t)$ and $g_i(t) = g_{i-1}(t)/(t - x_i)$.
- Or: put $A = H^*(X)[x_1, \dots, x_k]$ and $h(t) = \prod(t - x_i) \in A[t]$. By long division: $f_V(t) = h(t)q(t) + r(t)$ with $\deg(r(t)) < k$.
- Now $r(t) = m_0 + m_1 t + \dots + m_{k-1}t^{k-1}$ with $m_i \in A$, and $H^*(\mathrm{Flag}_k(V)) = A/(m_0, \dots, m_{k-1})$, so $f_V(t) = h(t)q(t)$ in $H^*(\mathrm{Flag}_k(V))[t]$.

Cohomology of flag bundles

- Let V be a d-dimensional complex vector bundle over X.
- Put $\mathrm{Flag}_k(V) = \{(x, W_0 < W_1 < \dots < W_k \le V_x) \mid \dim(W_i) = i\}$.
- Over $\mathrm{Flag}_k(V)$ we have line bundles $L_1, \dots, L_k$ with fibres $(L_i)_{(x,W)} = W_i/W_{i-1}$ and also a vector bundle U_k with $(U_k)_{(x,W)} = V_x/W_k$.
- We put $x_i = e(L_i) \in H^2(\mathrm{Flag}_k(V))$.
- Note that $L_1 \oplus \dots \oplus L_k \oplus U_k \simeq \pi^*(V)$ so $f_V(t) = f_{U_k}(t)\prod_{i=1}^k(t - x_i)$ in $H^*(\mathrm{Flag}_k(V))[t]$.
- A point of $P(U_{k-1})$ consists of a point $(x, W) = (x, W_0 < \dots < W_{k-1}) \in \mathrm{Flag}_{k-1}(V)$ together with a one-dimensional subspace $M \le (U_{k-1})_{(x,W)} = V_x/W_{k-1}$. This must have the form $M = W_k/W_{k-1}$ for a unique W_k with $W_{k-1} < W_k \le V_x$ and $\dim(W_k) = k$. Thus $\mathrm{Flag}_k(V) = P(U_{k-1})$.
- By the Projective Bundle Theorem: $H^*(\mathrm{Flag}_k(V)) = H^*(\mathrm{Flag}_{k-1}(V))\{x_k^i \mid 0 \le i \le d - k\}$.
- By induction: monomials $x_1^{i_1} \cdots x_k^{i_k}$ with $0 \le i_t \le d - t$ give a basis for $H^*(\mathrm{Flag}_k(V))$ over $H^*(X)$.
- In particular: these monomials give a basis for $H^*(\mathrm{Flag}_k(\mathbb{C}^d))$ over $\mathbb{Z}$.