

Multi-objective Controller Design:

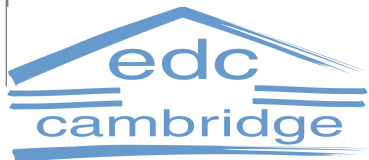
Evolutionary algorithms and Bilinear Matrix Inequalities for a passive suspension

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Linear Matrix Inequalities

A linear matrix inequality (LMI) is an expression,

$$F(x) = F_0 + x_1 F_1 + \cdots + x_n F_n > 0$$

- $F_i = F_i^T$ are real symmetric matrices,
- The inequality means that $F(x)$ is positive definite matrix

LMI example problem:

$$F(x_1, x_2) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} > 0$$

this is equivalent to

$$F(x_1, x_2) = \begin{bmatrix} 2 - x_1 & -(x_1 + x_2) & 0 \\ -(x_1 + x_2) & 5 - x_2 & x_2 \\ 0 & x_2 & -x_1 \end{bmatrix} > 0$$

Geometry of LMIs (Convexity)

$$F(x_1, x_2) = \begin{bmatrix} 2 - x_1 & -(x_1 + x_2) & 0 \\ -(x_1 + x_2) & 5 - x_2 & x_2 \\ 0 & x_2 & -x_1 \end{bmatrix} > 0$$

● **Feasible** iff all principal minors are nonnegative polynomial inequalities $f_i(x) > 0$

$$f_1(x) = 2 - x_1 > 0,$$

$$f_2(x) = 5 - x_2 > 0,$$

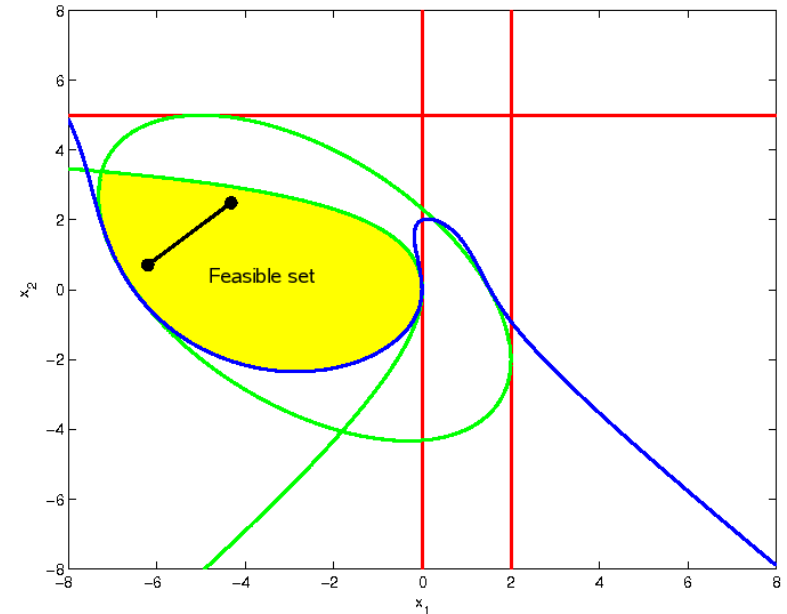
$$f_3(x) = -x_1 > 0,$$

$$f_4(x) = (2 - x_1)(5 - x_2) - (x_1 + x_2)^2 > 0,$$

$$f_5(x) = x_1(2 - x_1) > 0,$$

$$f_6(x) = -x_1(5 - x_2) - x_2^2 > 0,$$

$$f_7(x) = -x_1((2 - x_1)(5 - x_2) - (x_1 + x_2)^2) - x_2^2(2 - x_1) > 0$$



Intersection of LMIs

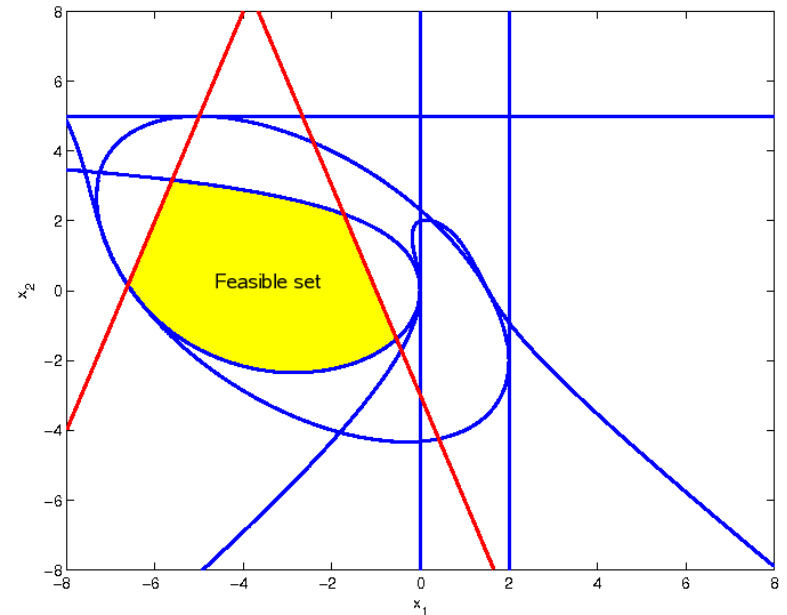
- If $F(x)$ and $G(x)$ are LMIs, the **intersection** has the form

$$H(x) = \begin{bmatrix} F(x) & 0 \\ 0 & G(x) \end{bmatrix} > 0$$

- **Example of intersections**

$$F(x) = \begin{bmatrix} 2 - x_1 & -(x_1 + x_2) & 0 \\ -(x_1 + x_2) & 5 - x_2 & x_2 \\ 0 & x_2 & -x_1 \end{bmatrix} > 0$$

$$G(x) = \begin{bmatrix} 3x_1 + x_2 + 3 & 0 \\ 0 & -3x_1 + x_2 - 20 \end{bmatrix} > 0$$

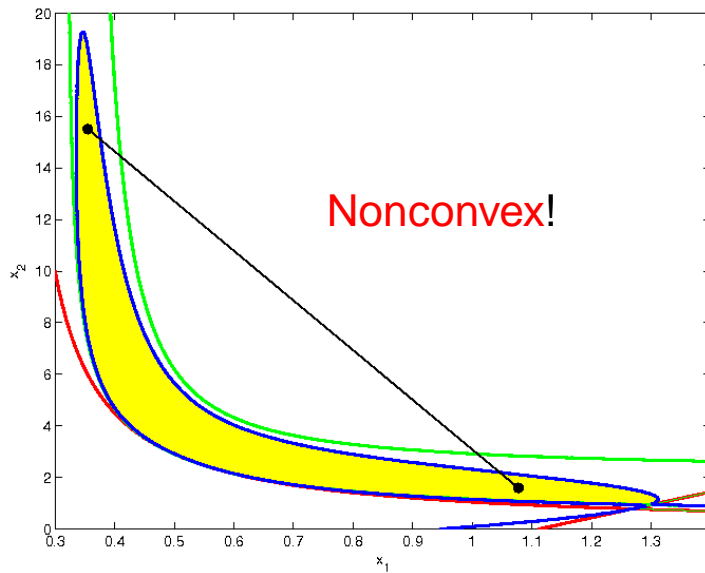


Bilinear Matrix Inequalities

A bilinear matrix inequality (BMI) is an expression,

$$F(x) = F_0 + \sum_i x_i F_i + \sum_i \sum_j x_i x_j F_{ij} < 0$$

BMI example (Henrion,2005):



$$F(x) = \begin{bmatrix} -10 & -0.5 & -2 \\ -0.5 & 4.5 & 0 \\ -2 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} 9 & 0.5 & 0 \\ 0.5 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix} +$$

$$x_2 \begin{bmatrix} -1.8 & -0.1 & -0.4 \\ -0.1 & 1.2 & -1 \\ -0.4 & -1 & 0 \end{bmatrix} + x_1 x_2 \begin{bmatrix} 0 & 0 & 2 \\ 0 & -5.5 & 3 \\ 2 & 3 & 0 \end{bmatrix} < 0$$

Mixed H_2/H_∞ Control

State-feedback close-loop system with two performance channels

$$\begin{aligned}\dot{x} &= (A + B_w K)x + B_w w \\ z_\infty &= (C_\infty + D_{\infty u} K)x + D_{\infty w} w \\ z_2 &= (C_2 + D_{2u} K)x\end{aligned}$$

and mixed performance specification

$$\|T_{w \rightarrow z_\infty}\|_\infty < \gamma_\infty \quad \text{and} \quad \|T_{w \rightarrow z_2}\|_2 < \gamma_2,$$

BMIs for the H_∞ norm constraint

$$\begin{bmatrix} AX_\infty + B_u KX_\infty + (*) + B_w B_w^T & * \\ C_\infty X_\infty + D_{\infty u} KX_\infty + D_{\infty w} B_w^T & D_{\infty w} D_{\infty w}^T - \gamma_\infty^2 I \end{bmatrix} < 0, \quad X_\infty > 0$$

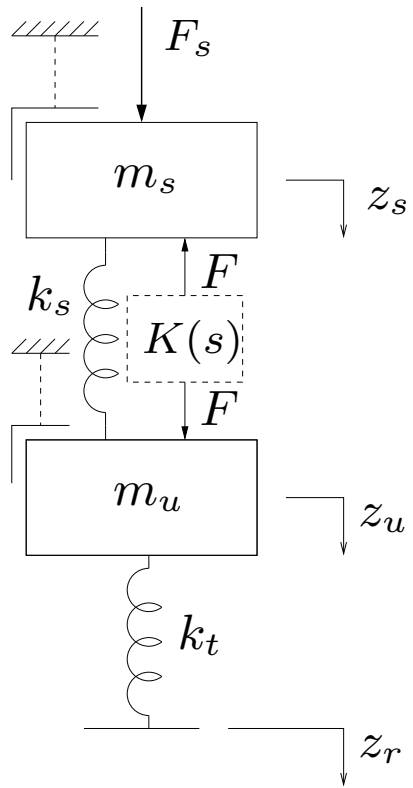
BMIs for the H_2 norm constraint

$$\begin{bmatrix} AX_2 + B_u KX_2 + (*) & B_w \\ * & -I \end{bmatrix} < 0, \quad \begin{bmatrix} W & C_2 X_2 + D_{2u} KX_2 \\ * & X_2 \end{bmatrix} > 0, \quad \text{tr}(W) < \gamma_2$$

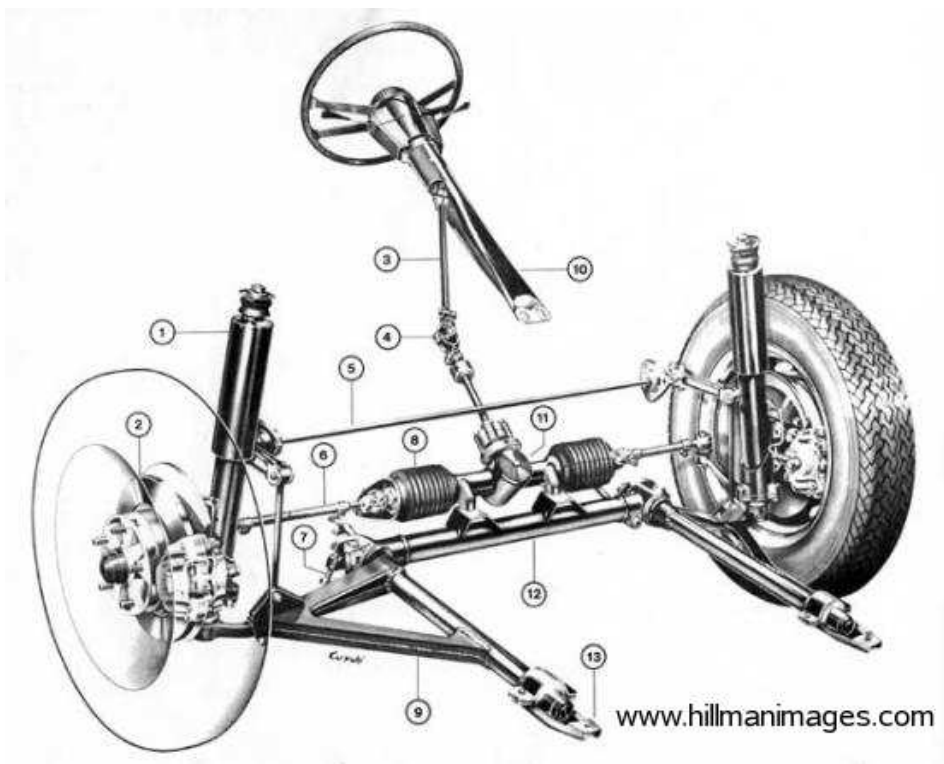
Two terms KX_∞ and KX_2 cannot be linearized simultaneously, Remedy $X = X_\infty = X_2$, then $Y = KX$ as change of variable to recover convexity, this leads to **conservative** results.

Mechanical network: vehicle suspension

quarter-car model



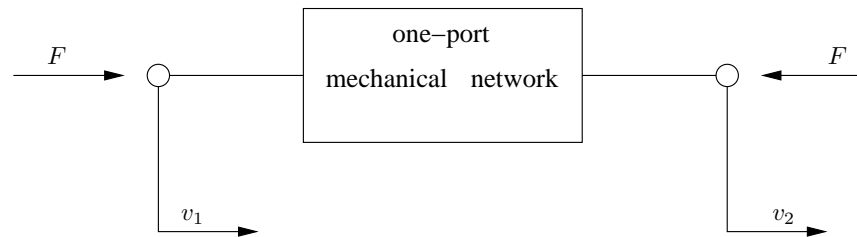
Front suspension



Mechanical networks

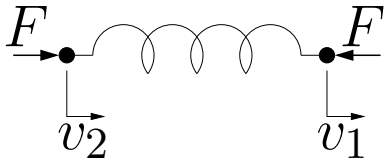
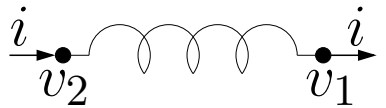
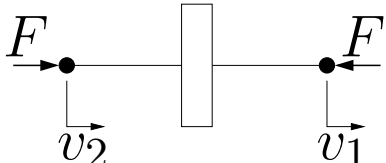
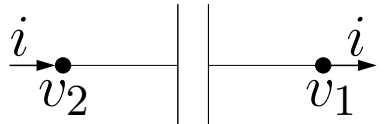
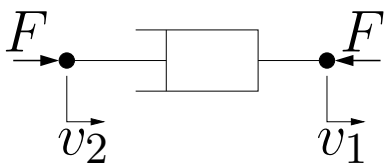

- A **mechanical network** is a rigid interconnection of mechanical elements (springs, masses, dampers and levers).
- A mechanical one-**port** network with force-velocity pair (F, v) is defined to be **passive** if,

$$\int_0^T F(t)v(t)dt \geq 0.$$



- **Impedance** is defined as a **positive-real** function $Z(s) = \frac{\text{across variable}}{\text{through variable}} = \frac{\hat{v}}{\hat{F}}$
- **Admittance** is defined as a **positive-real** function $Y(s) = \frac{1}{Z(s)} = \frac{\hat{F}}{\hat{v}}$.

Force-current analogy

Mechanical	Electrical
 $Y(s) = \frac{k}{s}$ $\frac{dF}{dt} = k(v_2 - v_1)$ <p>spring</p>	 $Y(s) = \frac{1}{Ls}$ $\frac{di}{dt} = \frac{1}{L}(v_2 - v_1)$ <p>inductor</p>
 $Y(s) = bs$ $F = b \frac{d(v_2 - v_1)}{dt}$ <p>mass?</p>	 $Y(s) = Cs$ $i = C \frac{d(v_2 - v_1)}{dt}$ <p>capacitor</p>
 $Y(s) = c$ $F = c(v_2 - v_1)$ <p>damper</p>	 $Y(s) = \frac{1}{R}$ $i = \frac{1}{R}(v_2 - v_1)$ <p>resistor</p>

The Inerter

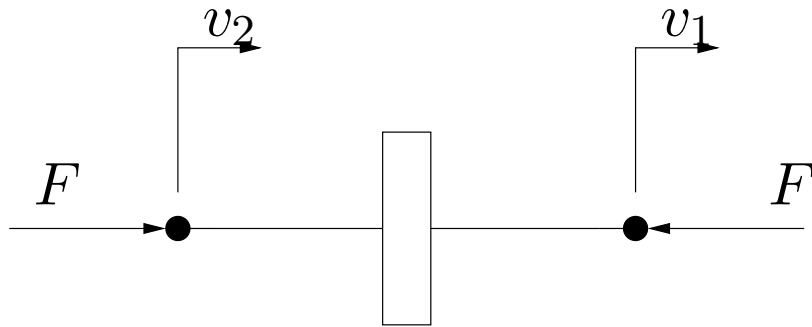
Concept: A mechanical **two-terminal** device such that the relative acceleration between the terminals is proportional to the force applied at the terminals.

- **M. C. Smith**, Synthesis of Mechanical Networks: The Inerter, IEEE Transactions on Automatic Control, 47(10), 1648 – 1662, 2002

The Inerter

Concept: A mechanical **two-terminal** device such that the relative acceleration between the terminals is proportional to the force applied at the terminals.

Theory:



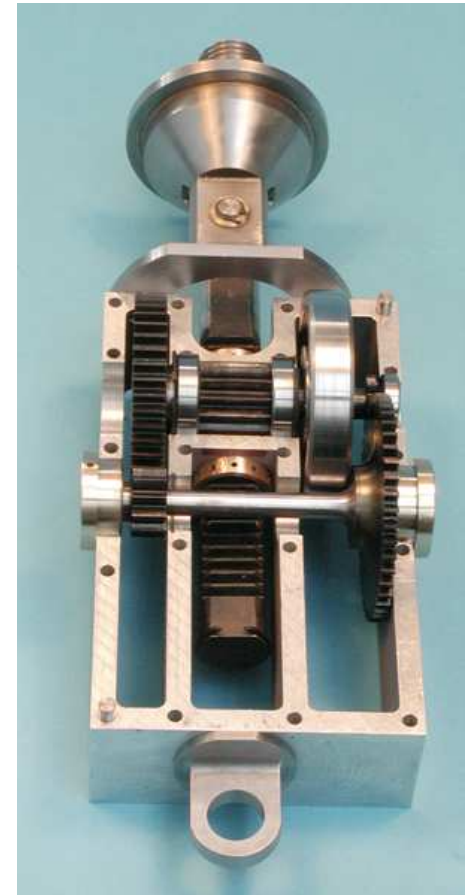
$$F = b(\dot{v}_2 - \dot{v}_1)$$

b : inertance

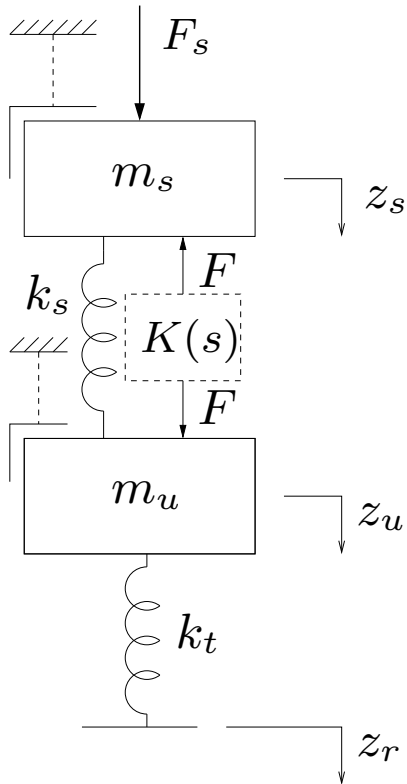
Practice: Can one build an inerter?

Rack-and-pinion inerter, mass ≈ 2 kg,

inertance = 70 – 700 kg, travel = 80 mm



Suspension for the quarter-car model



sprung and unsprung mass

$$\ddot{z}_s = \frac{F_s}{m_s} - \frac{F}{m_s} - \frac{k_s}{m_s}(z_s - z_u),$$

$$\ddot{z}_u = \frac{F}{m_u} + \frac{k_s}{m_u}(z_s - z_u) + \frac{k_t}{m_u}(z_r - z_u),$$

suspension

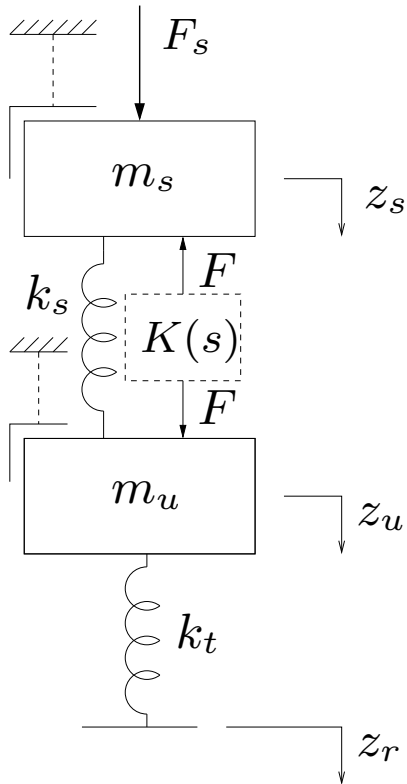
$$\hat{F} = K(s)(s\hat{z}_s - s\hat{z}_u),$$

Ride comfort:

$$J_1 := 2\pi(V\kappa)^{(1/2)} \|sT_{\hat{z}_r \rightarrow \hat{z}_s}\|_2$$

RMS body vertical acceleration in response to road disturbances

Suspension for the quarter-car model



sprung and unsprung mass

$$\ddot{z}_s = \frac{F_s}{m_s} - \frac{F}{m_s} - \frac{k_s}{m_s}(z_s - z_u),$$

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suspension

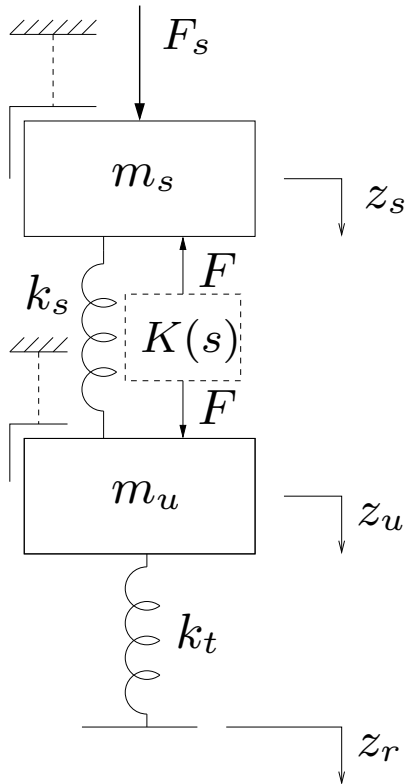
$$\hat{F} = K(s)(s\hat{z}_s - s\hat{z}_u),$$

Tyre grip:

$$J_3 := 2\pi(V\kappa)^{1/2} \left\| \frac{1}{s} T_{\hat{z}_r \rightarrow k_t(\hat{z}_u - \hat{z}_r)} \right\|_2$$

RMS dynamic tyre load in response to road disturbances

Suspension for the quarter-car model



sprung and unsprung mass

$$\ddot{z}_s = \frac{F_s}{m_s} - \frac{F}{m_s} - \frac{k_s}{m_s}(z_s - z_u),$$

$$\ddot{z}_u = \frac{F}{m_u} + \frac{k_s}{m_u}(z_s - z_u) + \frac{k_t}{m_u}(z_r - z_u),$$

suspension

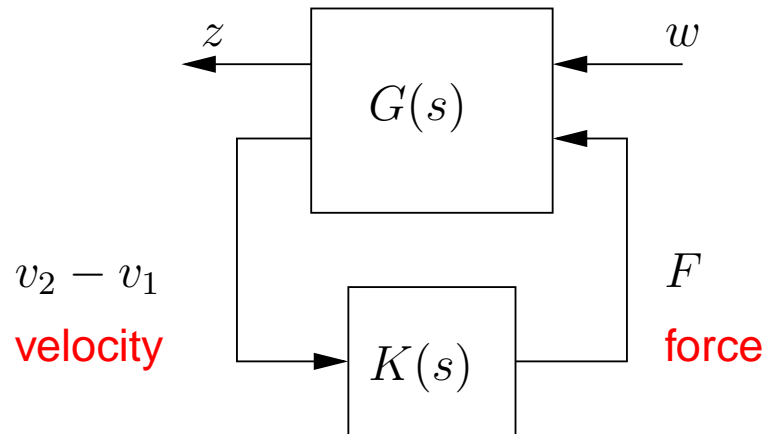
$$\hat{F} = K(s)(s\hat{z}_s - s\hat{z}_u),$$

Rejection of external loads:

$$J_5 := \left\| T_{\hat{F}_s \rightarrow \hat{z}_s} \right\|_{\infty}$$

Control problem formulation

- **Objective:** Synthesize a positive real admittance $Y(s)$ to improve performance criteria.
- **Formulation:** Optimize a vector of $\|T_{w \rightarrow z}\|$ over positive real controllers $K(s)$.



- **Solution:** Characterize and solve the problem using **Linear Matrix Inequalities**

LMI Formulation

- Characterize \mathcal{H}_∞ and \mathcal{H}_2 performances (Scherer et al., 1997),
 - $\|T_{w \rightarrow z}\|_2^2 = \text{Tr}(CXC^T)$, X solves a Lyapunov equation ...
 - Bounded real lemma of $\|T_{w \rightarrow z}\|_\infty = \sup_{w \in \mathcal{L}_2} \frac{\|z\|_2}{\|w\|_2}$...
- Positive Real Lemma (Boyd et al., 1994): Given that $K(s)$ positive real,

$$\exists X > 0, \begin{bmatrix} A^T X + XA & XB - C^T \\ B^T X - C & -D^T - D \end{bmatrix} \leq 0$$

multi-objective controller design

- Simultaneous J_1 and J_3 minimization,

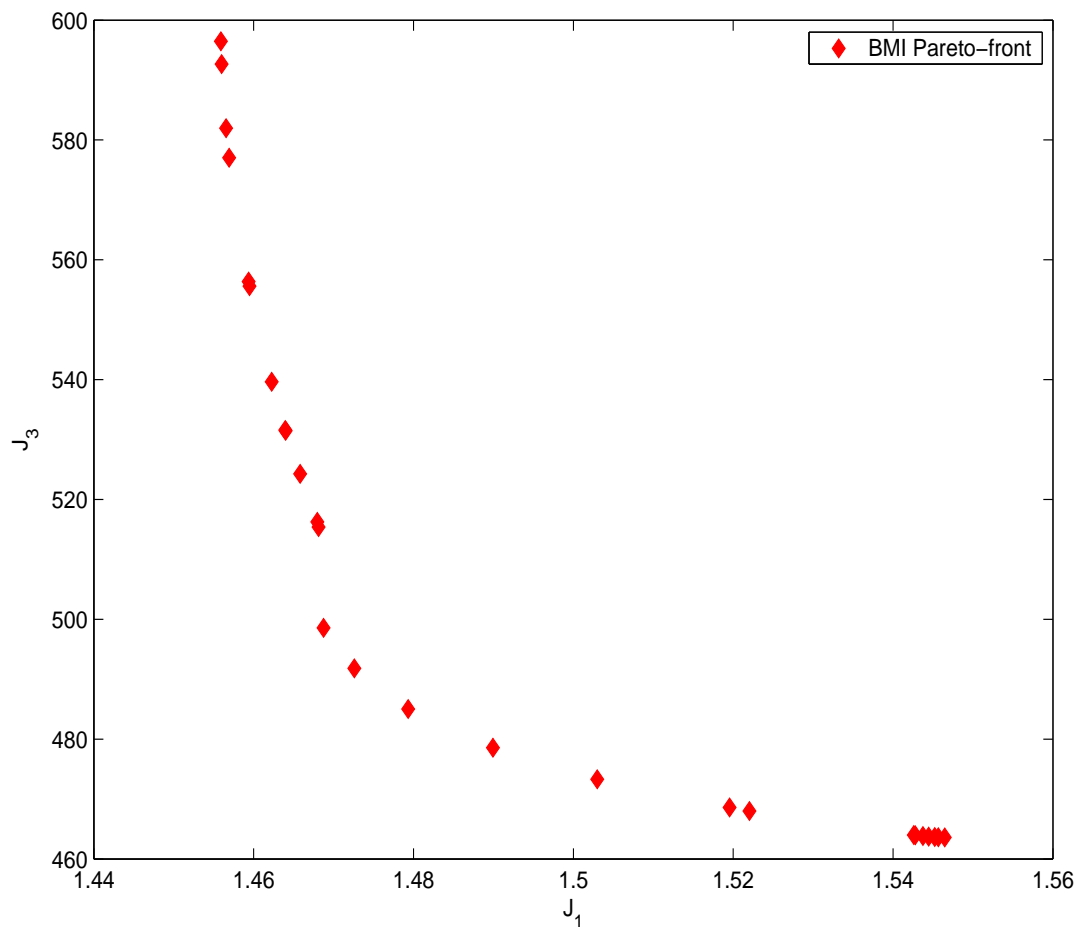
$$\min_{K(s) \text{ positive real}} \|T_{\hat{z}_r \rightarrow \hat{z}_s}\|_2 \text{ and } \|T_{\hat{z}_r \rightarrow f(\hat{z}_u - \hat{z}_r)}\|_2$$

- Positive real constraint \rightarrow **bilinear matrix inequality** with respect to Lyapunov matrices X_{cl} , X_k and controller matrix $K(s)$
- The approach taken here is to minimise

$$\sqrt{(1 - \lambda) \frac{J_1^2}{\hat{J}_1^2} + \lambda \frac{J_3^2}{\hat{J}_3^2}}, \text{ for } 0 < \lambda < 1$$

- Solved **locally** with **iterative** convex optimization methods

Multi-objective optimisation with BMIs



Lolcal Search: Relies on the intuitive choice of a feasible starting point

MOGA-based method

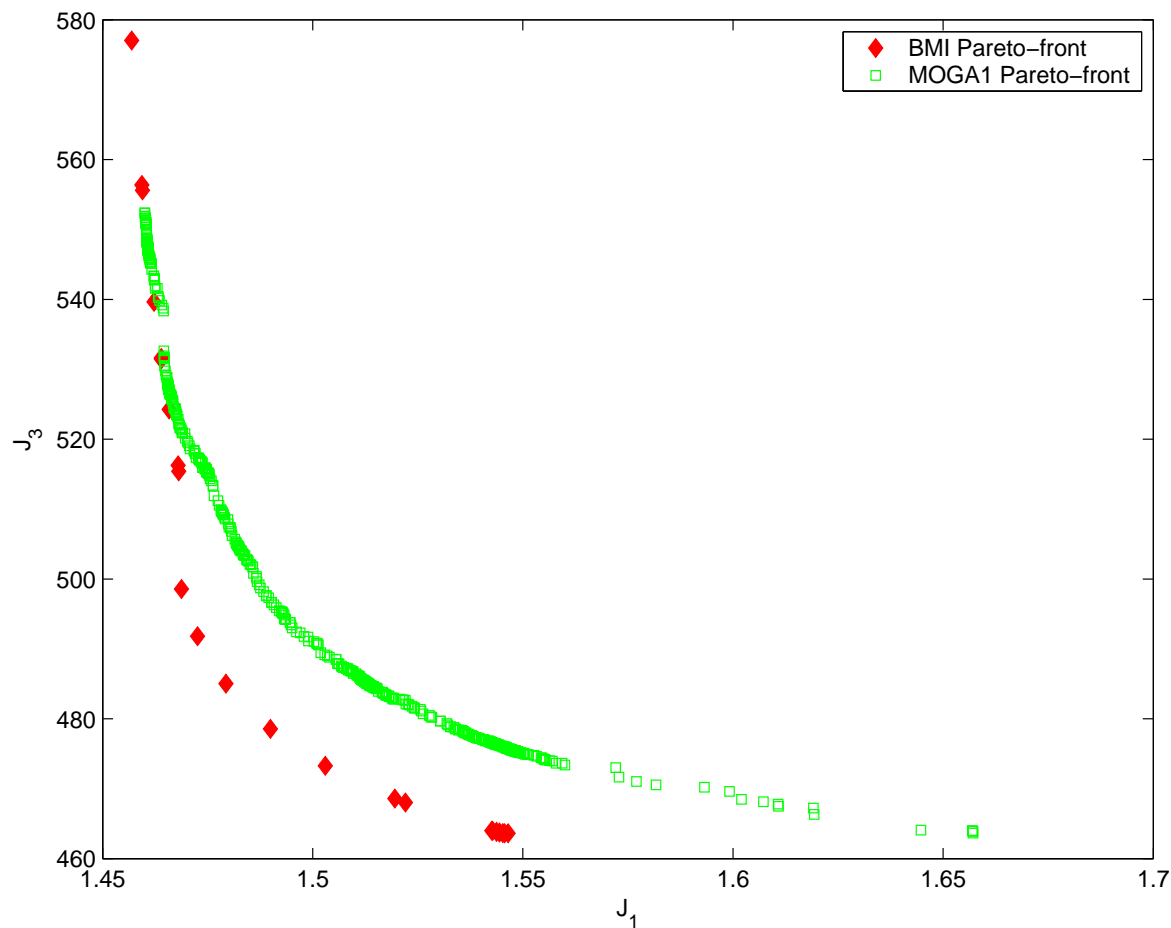
- **Parameter encoding:** The decision variables \rightarrow controller

$$K(s) = bs + c \frac{s^2 + a_1s + a_2}{s^2 + b_1s + b_2}$$

- **Optimise simultaneously:**

- Ride comfort $J_1 := 2\pi(V\kappa)^{(1/2)} \|sT_{\hat{z}_r \rightarrow \hat{z}_s}\|_2$
- Tyre grip $J_3 := 2\pi(V\kappa)^{1/2} \left\| \frac{1}{s} T_{\hat{z}_r \rightarrow k_t(\hat{z}_u - \hat{z}_r)} \right\|_2$
- subject to $K(s)$ been positive real (constraint)

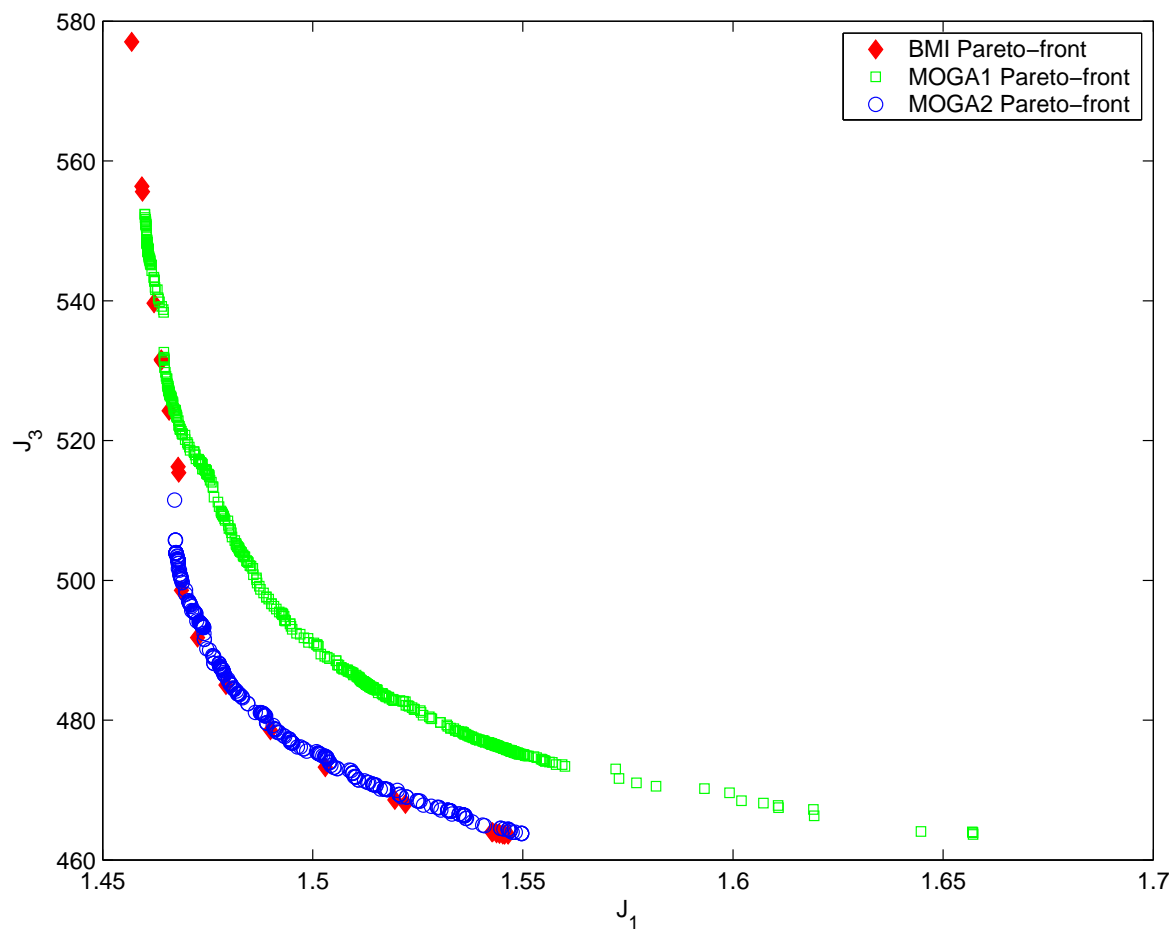
Multi-objective optimisation with GAs



Converge to a Local Pareto optimum

Deceptive problem: often the entire search favors the non-global

Multi-objective optimisation with GAs



Remedy: Increase the population size (from 200 to 600 individuals)

MOGA-based method: three objectives

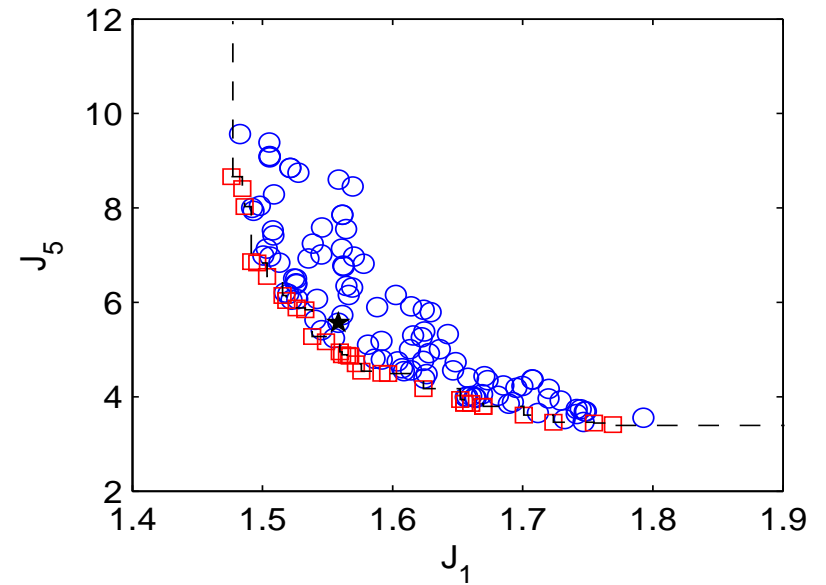
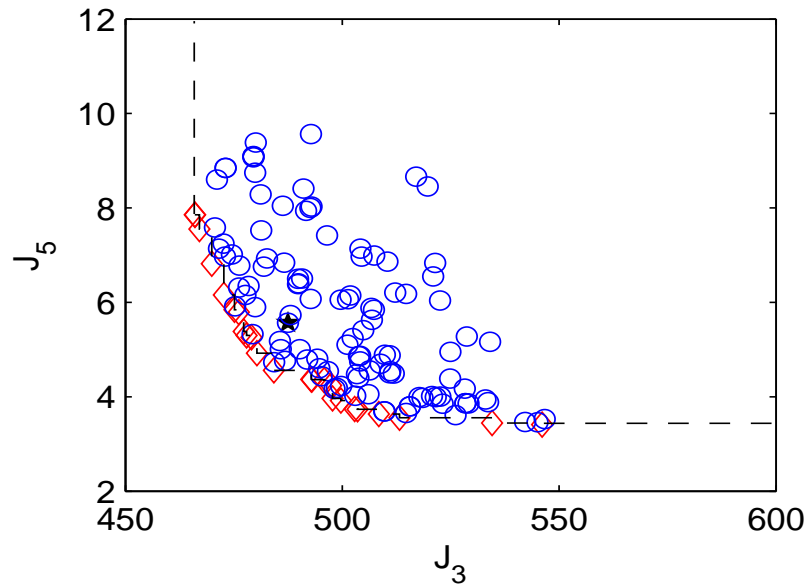
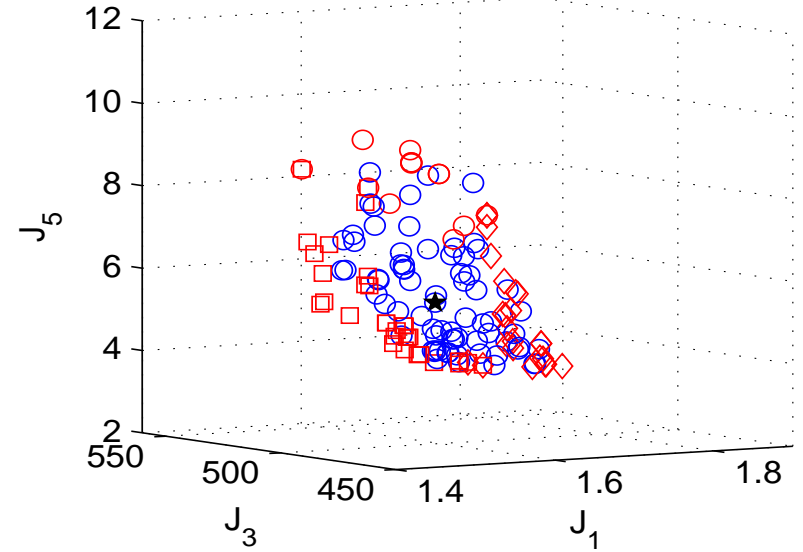
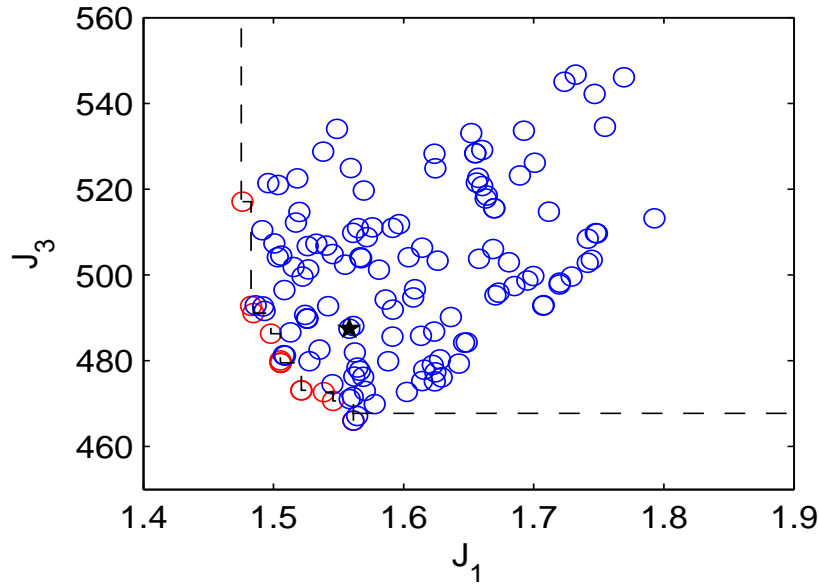
- **Parameter encoding:** The decision variables \rightarrow controller

$$K(s) = bs + c \frac{s^2 + a_1s + a_2}{s^2 + b_1s + b_2}$$

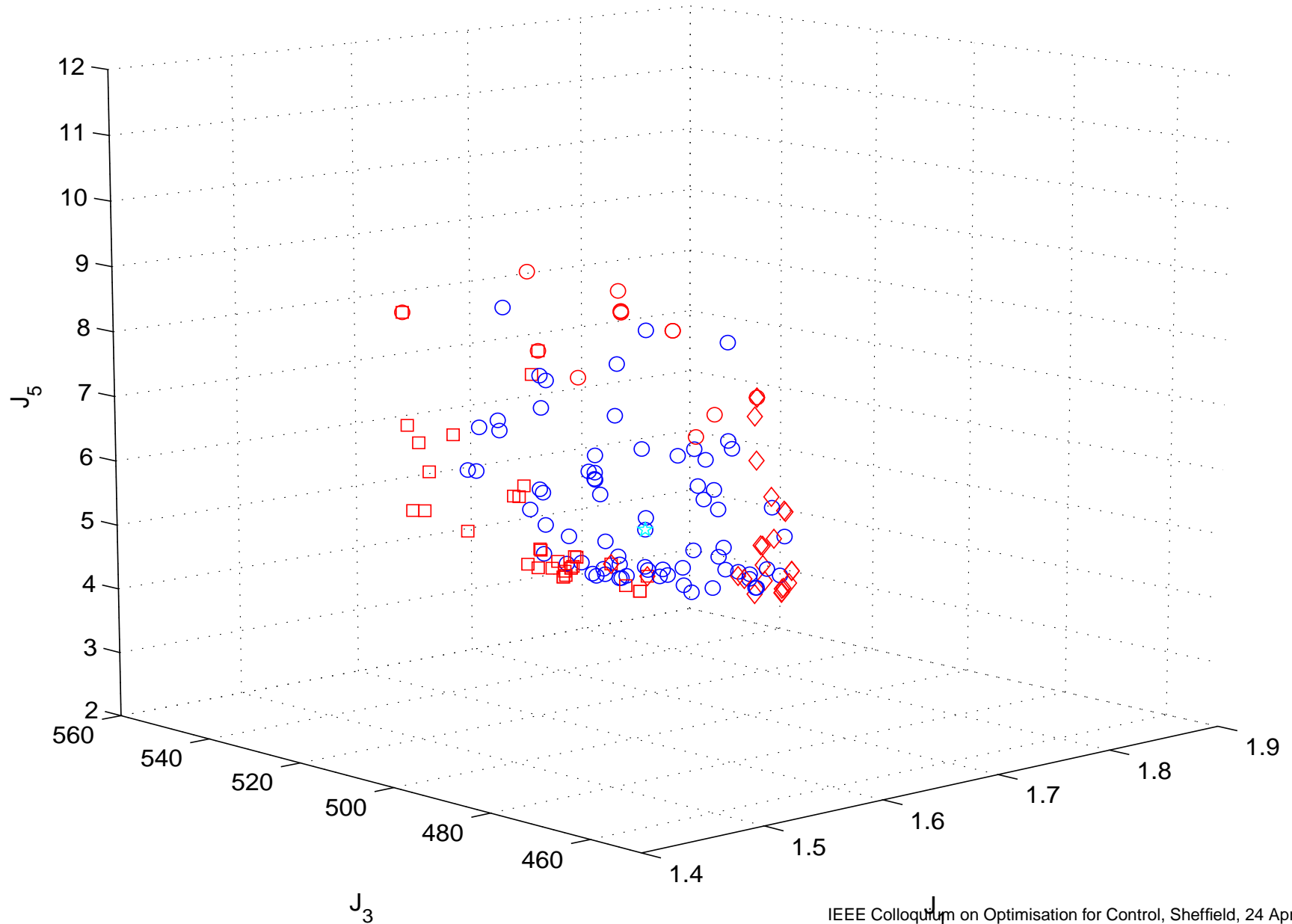
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- Ride comfort: $J_1 := 2\pi(V\kappa)^{(1/2)} \|sT_{\hat{z}_r \rightarrow \hat{z}_s}\|_2$
- Tyre grip: $J_3 := 2\pi(V\kappa)^{1/2} \left\| \frac{1}{s} T_{\hat{z}_r \rightarrow k_t(\hat{z}_u - \hat{z}_r)} \right\|_2$
- Rejection to external roads: $J_5 := \left\| T_{\hat{F}_s \rightarrow \hat{z}_s} \right\|_\infty$
- subject to $K(s)$ been positive real (constraint)

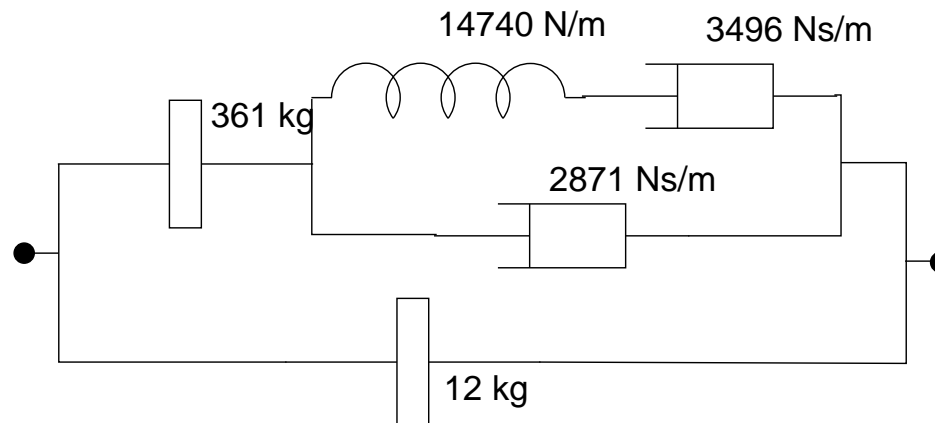
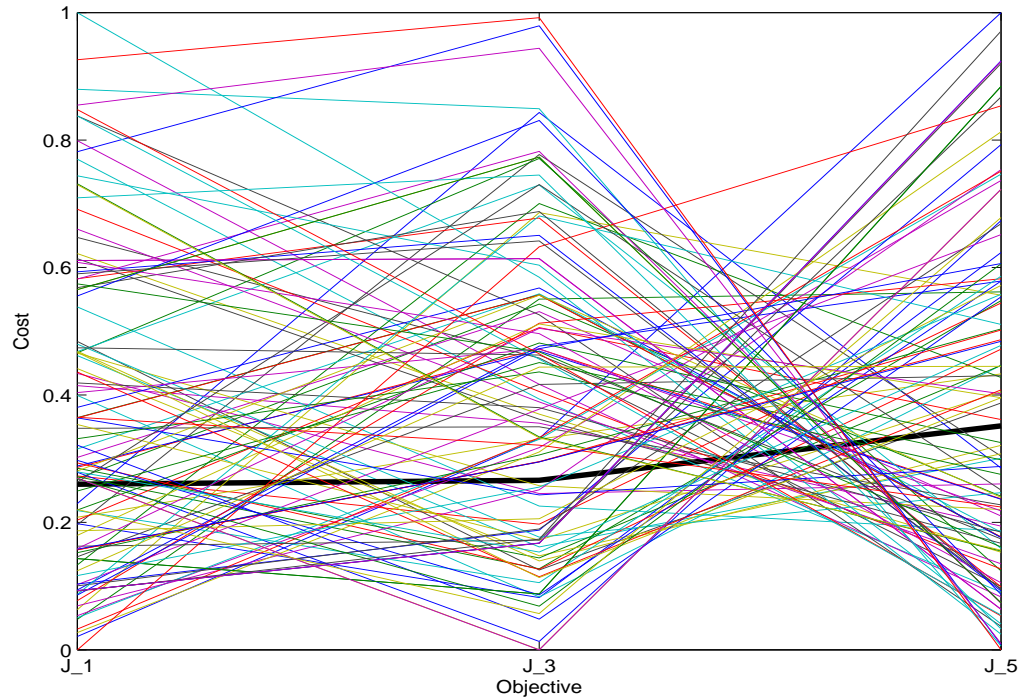
Three-objective optimisation



3D Pareto front



Realisation of the "best" suspension



Conclusions

- **BMI optimisation** over positive real controllers has been shown to be effective for two-objective optimisation problem. However, the design of a **single controller requires several attempts**.
- **MOGA-base method**- The Pareto-front can be investigated in a **single optimisation run**.
- **More than two objectives** can be included straightforwardly Thusm the designer engineer has a **choice from among the Pareto-optimal set**.
- A Possible **disadvantage of using MOGA** is that **some experience may be needed to choose** appropriate parameter values, such as the **population size**.

The end

Thank you,

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Reference:

A Molina-Cristobal, C Papageorgiou, G T Parks, M C Smith, P J Clarkson. Multi-objective Controller Design: Evolutionary Algorithms and Bilinear Matrix Inequalities for a Passive Suspension *Proceedings of the IFAC Workshop on Control Applications of Optimization, Cachan, France, April 2006*