Robust predictive control

Optimization over state feedback policies

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Outline

- Problem definition
  - LTI system with bounded state disturbance
  - Constraints on state and input
  - Minimize expected value of a quadratic cost function
- Optimization over affine **state** feedback policies
  - Optimization problem is **non-convex**
- Optimization over affine **disturbance** feedback policies
  - Equivalent to affine state feedback
  - Optimization problem is **convex**
- Guaranteeing stability for receding horizon control (RHC)

Optimization problem we would like to solve

At each time instant, solve

$$\inf_{\pi:=\{\mu_0(\cdot),\mu_1(\cdot),\ldots\}} \mathbb{E}\left[\sum_{i=0}^{\infty} x_i^T Q x_i + u_i^T R u_i\right]$$

where the measurement of the current state $x = x_0$,

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i + w_i, && i = 0, 1, \ldots \\ u_i &= \mu_i(x_0, \ldots, x_i), && i = 0, 1, \ldots \end{aligned}$$

Additionally, the **feedback** policy $\pi := \{\mu_0(\cdot), \mu_1(\cdot), \ldots\}$, where each $\mu_i(\cdot)$ is a control **law**, should ensure the constraints

$$h_i(x_i, u_i) \leq 0, \quad i = 0, 1, \ldots$$

are satisfied for any disturbance sequence $\{w_0, w_1, \ldots\}$, where each $w_i \in W$.

Optimization over feedback policies

Difficulties with solving the problem:

- Optimization is over **functions**: $\mu_i : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \rightarrow \mathbb{R}^m$
- Infinite number of decision variables: $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \ldots\}$
- Infinite number of constraints: $\forall \mathbf{w} \in \mathcal{W},\ h_i(x, \pi, \mathbf{w}) \leq 0$

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Towards a possible solution:

- Finite control horizon, i.e. $\pi = \{\mu_0(\cdot), \mu_1(\cdot), \ldots, \mu_{N-1}(\cdot)\}$
- Parameterize π in terms of a finite number of decision variables, e.g.

$$u_i = \mu_i(x_0, \ldots, x_i) = g_i + \sum_{j=0}^{i} L_{i,j} x_j$$

- Remove the universal quantifier from the description of constraints:

$$\forall \mathbf{w} \in \mathcal{W},\ h_i(x, \pi, \mathbf{w}) \leq 0 \iff h_i^*(x, \pi) := \sup\left\{h_i(x, \pi, \mathbf{w}) \mid \mathbf{w} \in \mathcal{W}\right\} \leq 0$$

New optimization problem

At each time instant, solve the optimization problem

$$\mathbb{P}_N(x): \qquad \inf_{\{g_i\},\{L_{i,j}\}} \mathbb{E}\left[x_N^T P x_N + \sum_{i=0}^{N-1} x_i^T Q x_i + u_i^T R u_i\right]$$

such that for all $\mathbf{w} := \begin{pmatrix} w_0^T & \cdots & w_{N-1}^T \end{pmatrix}^T \in \mathcal{W}$,

$$\begin{aligned}
x_{i+1} &= Ax_i + Bu_i + w_i, & i = 0, 1, \ldots, N-1 \\
u_i &= g_i + \sum_{j=0}^{i} L_{i,j} x_j, & i = 0, 1, \ldots, N-1 \\
(x_i, u_i) \in Z &:= \{(x, u) \mid Cx + Du \leq b\}, & i = 0, 1, \ldots, N-1 \\
x_N \in X_f &:= \{x \mid Yx \leq z\}
\end{aligned}$$

where the measurement of the current state is $x = x_0$ and $\mathcal{W} := W \times \cdots \times W$.

Some notation

Write feedback policy in matrix form:

$$\begin{pmatrix} u_0 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} L_{0,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ L_{N-1,0} & \cdots & L_{N-1,N-1} & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix} + \begin{pmatrix} g_0 \\ \vdots \\ g_{N-1} \end{pmatrix}$$

so that

$$\mathbf{u} = \mathbf{L}\mathbf{x} + \mathbf{g}$$

and let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{E}$ be matrices such that

$$\mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w} = (I - \mathbf{B}\mathbf{L})^{-1}(\mathbf{A}x + \mathbf{B}\mathbf{g} + \mathbf{E}\mathbf{w}),$$

where the current measurement of the state $x = x_0$.

Optimization problem rewritten

For a suitably-defined $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{D}$ and c, problem $\mathbb{P}_N(x)$ is equivalent to:

$$(\mathbf{L}^*(x), \mathbf{g}^*(x)) := \arg \inf_{(\mathbf{L},\mathbf{g})} \mathbb{E}\left[\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right]$$

subject to $\mathbf{L}$ block lower triangular and

$$\forall \mathbf{w} \in \mathcal{W}, \quad \mathbf{C}\mathbf{u} + \mathbf{D}\mathbf{x} \leq c, \quad \mathbf{u} = \mathbf{L}\mathbf{x} + \mathbf{g}, \quad \mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w}$$

The receding horizon control (RHC) law is defined as:

$$u = \kappa_N(x) := L^*_{0,0}(x)x + g^*_0(x)$$

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Punchline: Direct computation of $(L^*_{0,0}(x), g^*_0(x))$ is **difficult**, because the cost and constraint functions are **non-convex** in $(\mathbf{L}, \mathbf{g})$:

$$\mathbf{x} = (I - \mathbf{B}\mathbf{L})^{-1}(\mathbf{A}x + \mathbf{B}\mathbf{g} + \mathbf{E}\mathbf{w}), \quad \mathbf{u} = \mathbf{L}(I - \mathbf{B}\mathbf{L})^{-1}(\mathbf{A}x + \mathbf{B}\mathbf{g} + \mathbf{E}\mathbf{w}) + \mathbf{g}$$

Affine disturbance feedback policies

Note that $w_i = x_{i+1} - Ax_i - Bu_i$. Consider parameterizing the control as an affine function of prior **disturbances**:

$$u_i = v_i + \sum_{j=0}^{i-1} M_{i,j} w_j, \quad i = 0, \dots, N-1$$

Write this in matrix form:

$$\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ M_{1,0} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ M_{N-1,0} & \cdots & M_{N-1,N-2} & 0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{N-1} \end{pmatrix} + \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{N-1} \end{pmatrix}$$

or

$$\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}$$

Equivalence between feedback policies

Affine **state** feedback and affine **disturbance** feedback are equivalent:

- Given any initial state x, the input and state trajectories due to state and disturbance feedback can be made to be equal for all allowable disturbance sequences:
  - Given any $(\mathbf{L}, \mathbf{g})$, there exists a pair $(\mathbf{M}, \mathbf{v})$ such that $\mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v} = \mathbf{L}\mathbf{x} + \mathbf{g}$ for all $\mathbf{w} \in \mathcal{W}$
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Sketch of proof:

$$\mathbf{M} = \mathbf{L}(I - \mathbf{B}\mathbf{L})^{-1}\mathbf{E}$$

$$\mathbf{v} = \mathbf{L}(I - \mathbf{B}\mathbf{L})^{-1}(\mathbf{A}x + \mathbf{B}\mathbf{g}) + \mathbf{g}$$

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Sketch of proof:

$$\mathbf{L} = (I - \mathbf{M}\mathbf{E}^{\dagger}\mathbf{B})^{-1}\mathbf{M}\mathbf{E}^{\dagger}$$

$$\mathbf{g} = (I - \mathbf{M}\mathbf{E}^{\dagger}\mathbf{B})^{-1}(\mathbf{v} - \mathbf{M}(\mathbf{A}^{\dagger})^{T}Ax)$$

Equivalent, but convex optimization problem

For a suitably-defined $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{D}$ and c, problem $\mathbb{P}_N(x)$ is equivalent to:

$$(\mathbf{M}^*(x), \mathbf{v}^*(x)) := \arg \inf_{(\mathbf{M},\mathbf{v})} \mathbb{E}\left[\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{u}^T\mathbf{R}\mathbf{u}\right]$$

subject to $\mathbf{M}$ *strictly* block lower triangular and

$$\forall \mathbf{w} \in \mathcal{W}, \quad \mathbf{C}\mathbf{u} + \mathbf{D}\mathbf{x} \leq c, \quad \mathbf{u} = \mathbf{M}\mathbf{w} + \mathbf{v}, \quad \mathbf{x} = \mathbf{A}x + \mathbf{B}\mathbf{u} + \mathbf{E}\mathbf{w}$$

The receding horizon control (RHC) law is now given by

$$u = \kappa_N(x) := L_{0,0}^*(x)x + g_0^*(x) = v_0^*(x)$$

Punchline: Direct computation of $v_0^*(x)$ is easy if $\mathcal{W}$ is convex, because the cost and constraint functions are **convex** in $(\mathbf{M}, \mathbf{v})$!

Convexity of constraints

For a given initial state x, one can rewrite the input and state constraints as:

$$F\mathbf{v} + (F\mathbf{M} + G)\mathbf{w} \le c + Hx, \quad \forall \mathbf{w} \in \mathcal{W},$$

where F, G, c and H are appropriately defined.

Remove the universal quantifier to get a set of finite and **tractable** constraints:

$$F\mathbf{v} + \sup_{\mathbf{w} \in \mathcal{W}} (F\mathbf{M} + G)\mathbf{w} \le c + Hx,$$

where the supremum is taken row-wise.

As an example, note that

$$W := \{w \mid \|w\|_\infty \le \eta\} \Rightarrow \sup_{w \in W} a^T w = \eta \|a\|_1$$

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As an example, note that

$$\mathcal{W} := \{\mathbf{w} \mid \|\mathbf{w}\|_\infty \leq \eta\} \Rightarrow \sup_{\mathbf{w} \in \mathcal{W}} (F\mathbf{M} + G)\mathbf{w} = \eta \, \mathrm{abs}(F\mathbf{M} + G)\mathbf{1}$$

Hence, the set $\{(\mathbf{M}, \mathbf{v}) \mid F\mathbf{v} + \eta \, \mathrm{abs}(F\mathbf{M} + G)\mathbf{1} \leq c + Hx\}$ is **convex**

Convexity of optimization problem

Without loss of generality, assume that

$$\mathbb{E}[w] = 0 \text{ and } W := \{w \mid \|w\|_\infty \leq \eta\}$$

$\mathbb{P}_N(x)$ is equivalent to a **tractable** and **convex** optimization problem:

$$(\mathbf{M}^*(x), \mathbf{v}^*(x)) = \arg \inf_{(\mathbf{M},\mathbf{v})} (\mathbf{A}x + \mathbf{B}\mathbf{v})^T \mathbf{Q}(\mathbf{A}x + \mathbf{B}\mathbf{v}) + \mathbf{v}^T \mathbf{R}\mathbf{v}$$

where the optimization is subject to $\mathbf{M}$ *strictly* block lower triangular and

$$F\mathbf{v} + \eta \,\mathrm{abs}(F\mathbf{M} + G)\mathbf{1} \leq c + Hx$$

This is because:

- $\mathbf{u}$ and $\mathbf{x}$ are linear in $\mathbf{w}$
- We have perfect state information
- The cost is quadratic and the expectation operator is linear

Conditions for stability

- Choose a terminal control law $u = Kx$ such that $A + BK$ stable
- Let $X_K := \{x \mid Cx + DKx \leq b\}$
- Compute a terminal constraint $X_f \subseteq X_K$ such that X_f is **robust positively invariant** for $x_{k+1} = (A + BK)x_k + w_k$, i.e.

$$(A + BK)x + w \in X_f, \quad \forall x \in X_f, w \in W$$

- Choose $Q \succ 0$, $R \succ 0$ and $P \succ 0$ such that the terminal cost $V_f(x) := x^T Px$ is a Lyapunov function for the **undisturbed** system $x_{k+1} = (A + BK)x_k$ in the sense that

$$V_f((A + BK)x) - V_f(x) \leq -x^T (Q + K^T RK)x, \quad \forall x \in X_f$$

NB: Similar assumptions as in conventional RHC, but proof of stability is more involved

Guaranteeing stability

If W is a polytope, then the closed-loop RHC system

$$x_{k+1} = Ax_k + B\kappa_N(x_k) + w_k, \quad w_k \in W$$

is input-to-state stable (ISS):

$$\|x_{k+1}\| \leq \beta\left(\|x_0\|, k\right) + \gamma\left(\sup_{i\in\{0,\ldots,k\}} \|w_i\|\right)$$

where $\beta(\cdot)$ is a $\mathcal{KL}$-function (continuous, non-negative and increasing in first argument, decreasing to zero in second argument) and $\gamma(\cdot)$ is a $\mathcal{K}$-function (continuous, non-negative and increasing)

The input and state constraints are satisfied for all time, given any disturbance sequence $\{w_0, w_1, \ldots\}$, where each $w_i \in W$.

Conclusions and remarks

- Affine **state** feedback policies
  - Optimization problem is **non-convex**
- Affine **disturbance** feedback policies
  - Equivalent to affine state feedback
  - Optimization problem is **convex** and **tractable**
  - Solve a QP if all sets are polytopic
  - Number of decision variables and constraints is $\mathcal{O}(N^2)$
- Receding horizon control
  - Choose appropriate terminal control law, cost and constraint
  - Input-to-state stable (ISS)
  - Guarantee constraint satisfaction for all time
- Can exploit structure for efficient computation
  - Sparse interior point code guarantees solution in time $\mathcal{O}(N^3)$
- Can use other cost functions, such as H_∞ control / ℓ_2 gain minimization